Import Networks, Fuzzy Relations and Semirings

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Abstract. In this work we give a brief sketch how to handle circulations in import networks with fuzzy relations. We use techniques developed by Kawahara in [Kaw] and modified in [Glü].

1 Introduction

Circulations in networks are similar to network flows, with two differences: there is no flow from a source to a sink, and at every node of the network a certain amount of flow is pumped into the network or has to leave it, cf. [Jun], section 7.7 in a slightly different context.

In this work chapter two gives brief introduction in the theory of fuzzy relations. In chapter three networks, flows and circulations are introduced, and chapter four shows a small piece of application of these constructs.

The aim is to give an algebraic approach to this problem, with the perspective of automated reasoning in this area.

2 Fuzzy Relations

2.1 Definitions and Basic Operations

Definition 2.1. A *fuzzy relation* α between sets X and Y, written $\alpha : X \leftrightarrow Y$, is a mapping from $X \times Y$ into the interval $[0, 1]$. A fuzzy Relation between a set X and itself is called a *fuzzy endorelation.* A fuzzy relation with a range contained in $[0, 1]$ is called Boolean.

The *empty relation* $\mathbf{0}_{XY}$, the *universal relation* ∇_{XY} and the *identity relation* id_X are given by $\mathbf{0}_{XY}(x, y) = 0$, $\nabla_{XY}(x, y) = 1$ for all $(x, y) \in X \times Y$ and $id_X(x, y) = \delta_{x,y}$ for $(x, y) \in X \times X$. A fuzzy relation with a range contained in $\{0, 1\}$ is called *boolean.*

For $a, b \in [0, 1]$ we define $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$, $a \ominus b = \max\{0, a - b\}$ and $a \oplus b = \min\{1, a + b\}$.

Consistent with these definitions we define the operators $\bigvee_{x\in X} x$ and $\bigwedge_{x\in X} x$ for arbitrary subsets $X \subseteq [0, 1]$ by $\bigvee_{x\in X} x := sup\{x \in X\}$ and $\bigwedge_{x\in X} x := inf\{x \in X\}$.

For fuzzy relations $\alpha, \beta : X \leftrightarrow Y$ the join $\alpha \sqcup \beta$, the meet $\alpha \sqcap \beta$, the truncating difference $\alpha \ominus \beta$ and the truncating sum $\alpha \oplus \beta : X \leftrightarrow Y$ are the pointwise extensions of the operators defined above: $(\alpha \sqcup \beta)(x, y,) = \alpha(x, y) \vee \beta(x, y)$, $(\alpha \sqcap \beta)(x, y,) = \alpha(x, y) \wedge \beta(x, y)$, $(\alpha \ominus \beta)(x, y,) = \alpha(x, y) \ominus \beta(x, y)$ and $(\alpha \oplus \beta)(x, y,) = \alpha(x, y) \oplus \beta(x, y)$.

Two fuzzy relations $\alpha, \beta : X \leftrightarrow Y$ are said to be *disjoint* if $\alpha \sqcap \beta = \mathbf{0}$. We abbreviate the union of two disjoint fuzzy relations α and β by $\alpha \dot{\sqcup} \beta$.

For two fuzzy relations $\alpha, \beta : X \leftrightarrow Y$ we write $\alpha \sqsubseteq \beta$, if $\alpha(x, y) \leq \beta(x, y)$ holds for all $(x, y) \in X \times Y$.

It is easy to see that meet, join and truncating sum are commutative and associative. Moreover, join distributes over meet and vice versa.

Another important operation on fuzzy relations is *scalar multiplication*. For real numbers $k \in [0, 1]$ and a fuzzy relation $\alpha : X \leftrightarrow Y$ the scalar product $k\alpha$, also written $k \cdot \alpha$, is defined by $(k\alpha)(x, y) = k \cdot (\alpha(x, y))$ for all $(x, y) \in X \times Y$. It distributes over $\sqcup, \sqcap, \ominus$ and $\oplus$.

For a fuzzy relation $\alpha : X \leftrightarrow Y$ the *converse* $\alpha^\sharp : Y \leftrightarrow X$ is defined by $\alpha^\sharp(y, x) = \alpha(x, y)$. It commutes with scalar multiplications and distributes over $\sqcup, \sqcap, \ominus$ and $\oplus$.

We define the *composition* $\alpha\beta : X \leftrightarrow Z$ of two fuzzy relations $\alpha : X \leftrightarrow Y$ and $\beta : Y \leftrightarrow Z$ by $\alpha\beta(x, z) = \bigvee_{y \in Y}(\alpha(x, y) \wedge \beta(y, z))$.

The composition of fuzzy relations distributes over join, i.e., $\alpha(\beta \sqcup \gamma) = \alpha\beta \sqcup \alpha\gamma$ and $(\alpha \sqcup \beta)\gamma = \alpha\gamma \sqcup \beta\gamma$. In general the composition does not distribute over meet, but for for univalent α the equality $\alpha(\beta \sqcap \gamma) = \alpha\beta \sqcap \alpha\gamma$ and for injective γ the equality $(\alpha \sqcap \beta)\gamma = \alpha\gamma \sqcap \beta\gamma$ hold.

Composition commutes with scalar multiplication, i.e., $k(\alpha\beta) = (k\alpha)\beta = \alpha(k\beta)$. Finally, composition is contravariant w.r.t. converse, i.e., $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$

The *n-th power* α^n of a fuzzy endorelation $\alpha : X \leftrightarrow X$ is defined inductively by $\alpha^0 = id_X$ and $\alpha^{n+1} = \alpha\alpha^n$ for $n \in \mathbb{N}_0$. The *reflexive and transitive closure* α^* of a fuzzy endorelation is defined by $\alpha^* = \bigvee_{n \in \mathbb{N}_0}$.

An important concept in the one of test relations:

Definition 2.2. *A Boolean subrelation of id_X is called a test relation on X.*

A subset X' of X corresponds to a test relation $\tau(X)$ on X, given by $\tau(X)(x, y) = 1$ if $x = y$ and $x \in X'$ and $\tau(X') = 0$ otherwise. Contrary every test relation on X describes a subset X' of X in an obvious manner. So we will use test relations to characterise subsets. For a single element $x \in X$ we abbreviate the corresponding test relation simply by x. Such a test relation is also called a *point relation*.

For every test relation τ an unique test relation τ^c, the *complement* of τ, exists, characterised by $\tau\tau^c = \mathbf{0}_{XX}$ and $\tau \sqcup \tau^c = id_X$. For a test relation $\tau(X')$ its complement is $\tau(\overline{X'})$.

To change the range (and hence the domain) of a fuzzy endorelation we introduce two operations, the embedding and the projection:

Definition 2.3. For a fuzzy endorelation $\alpha : X \leftrightarrow X$ on X and a superset $\lceil X \rceil$ of X, the *embedding* $\lceil \alpha \rceil$ of α into $\hat{X}$ is $\lceil \alpha \rceil : \hat{X} \leftrightarrow \hat{X}$ is given by $\lceil \alpha \rceil(x, y) = \alpha(x, y)$ for all $(x, y) \in X \times X$ and $\lceil \alpha \rceil(x, y) = 0$ otherwise.

The embedding of fuzzy relations distributes over join, meet, truncating sum, truncating difference and composition, i.e., $\lceil \alpha \circ \beta \rceil = \lceil \alpha \rceil \circ \lceil \beta \rceil$ for $\circ \in \{\sqcup, \sqcap, \oplus, \ominus, \cdot\}$. In particular we have $\lceil \alpha \rceil = \lceil id \rceil \lceil \alpha \rceil = \lceil \alpha \rceil \lceil id \rceil$. Embedding also commutes with converse, i.e., $\lceil \alpha^\sharp \rceil = (\lceil \alpha \rceil)^\sharp$. Moreover, it is order-preserving: $\alpha \sqsubseteq \beta$ implies $\lceil \alpha \rceil \sqsubseteq \lceil \beta \rceil$.

The dual operation to embedding is projection:

Definition 2.4. For a fuzzy endorelation $\alpha : \hat{X} \leftrightarrow \hat{X}$ its *projection* $\lfloor \alpha \rfloor : X \leftrightarrow X$ for a subset $X \subseteq \hat{X}$ is given by $\lfloor \alpha \rfloor(x, y) = \alpha(x, y)$ for all $x, y \in X$.

The projection has algebraic properties dual to the ones of the embedding. It is also order preserving, it commutes with the converse and distributes over join, meet, truncating sum and truncating difference, but in general not over composition.

For a test τ both the embedding $\lceil \tau \rceil$ and the projection $\lfloor \tau \rfloor$ are tests, too.

2.2 Cardinality of Fuzzy Relations

Definition 2.5. The *cardinality* $|\alpha|$ of a fuzzy relation $\alpha : X \leftrightarrow Y$ is defined by $|\alpha| = \sum_{(x,y) \in X \times Y} \alpha(x, y)$.

Obvious properties of the cardinality are:

- $|\alpha| \geq 0$
- $|\alpha| = 0 \Leftrightarrow \alpha = 0_{XY}$
- $|\alpha| = |\alpha^\sharp|$
- Cardinality is an isotone function, i.e., $\alpha \sqsubseteq \beta$ implies $|\alpha| \leq |\beta|$.

Because we will limit ourselves to fuzzy relations over finite sets the cardinality will always be a nonnegative real number.

A fuzzy relation $\alpha : X \leftrightarrow Y$ is called *normalised* if $|\alpha| \leq 1$. This implies $|\beta| \leq 1$ for all fuzzy relations β with $\beta \sqsubseteq \alpha$.

A connection between the meet and the cardinality is the equality

$$|\alpha \sqcup \beta| = |\alpha| + |\beta| - |\alpha \sqcap \beta| \tag{1}$$

for arbitrary fuzzy relations $\alpha, \beta : X \leftrightarrow Y$. In particular, it states $|\alpha \sqcup \beta| = |\alpha| + |\beta|$ for disjoint fuzzy relations α and β.

For fuzzy relations α, β, γ with $\beta \sqsubseteq \alpha$ and $\gamma \sqsubseteq \alpha \ominus \beta$ the equality $|\gamma| + |\beta| = |\gamma \oplus \beta|$ holds. If $\beta \sqsubseteq \alpha$ then $|\alpha \ominus \beta| = |\alpha| \ominus |\beta|$.

If $\alpha \sqsubseteq 0.5id$ and $\beta \sqsubseteq 0.5id$ hold then $|\alpha \oplus \beta| = |\alpha| + |\beta|$.

3 Flows and Circulations

Definition 3.1. For a finite set X of nodes, an s-t-network N is a triple $N = (\alpha : X \leftrightarrow X, s, t)$ where s and t are two distinct elements of X and α satisfies $\alpha \sqcap \alpha^\sharp = \mathbf{0}_{XX}$. If the condition $\alpha \sqcap \alpha^\sharp$ is dropped N is called s *pseudo-s-t-network*. An *import network* is a triple $I = (\alpha : X \leftrightarrow X, \iota : X \leftrightarrow X, \omega : X \leftrightarrow X)$ with $\alpha \sqcap \alpha^\sharp = \mathbf{0}_{XX}$, $\alpha, \iota, \omega \sqsubseteq 0.5id_X$ and $\iota \sqcap \omega = \mathbf{0}_{XX}$. An *import pseudonetwork* as defined analogously as above.

In both cases α is called the *capacity constraint*, s is the *source*, t is the *sink*, ι is the *import* and *omega* is the *output*.

Definition 3.2. A *flow* on an s-t-(pseudo)network $N = (\alpha : X \leftrightarrow X, s, t)$ is a fuzzy relation $\varphi : X \leftrightarrow X$ with $\varphi \sqsubseteq \alpha$ and $|\tau\varphi| = |\varphi\tau|$ for all test relations $\tau \sqsubseteq (s \sqcup t)^c$. A *circulation* φ on a import (pseudo)network $I = (\alpha : X \leftrightarrow X, \iota : X \leftrightarrow X, \omega : X \leftrightarrow X)$ is a fuzzy endorelation on X with $\varphi \sqsubseteq \alpha$ and $|(\varphi \oplus \iota)\tau| = |\tau(\varphi \oplus \omega)|$.

So a flow and a circulation have to respect the capacity constraint, the other two conditions are called *flow conservation*. The *value* $val(\varphi)$ of a flow φ in an s-t-network is given by $val(\varphi) = |s\varphi|$. It hold the equalities $val(\varphi) = |s\varphi| = \varphi t$.

A *cut* τ in an s-t-network is a test relation τ with $s \sqsubseteq \tau \sqsubseteq t^c$. The *capacity* $c(\tau)$ of a cut τ is defined via $c(\tau) = \tau\alpha\tau^c$.

A flow φ on a s-t-network N is called *maximal* if $val(\varphi) \geq val(\psi)$ holds for all flows ψ on N. A maximal always exists, contrary to a circulation. The famous Max Flow-Min Cut-Theorem states that the value of a maximal flow equals the minimal capacity of a cut in an s-t-network. Such a cut τ is said to be saturated by the maximal flow φ with the consequence that $\tau\alpha\tau^c = \tau\varphi\tau^c$ holds.

4 Determining Circulations

After these preliminaries we apply our tools to import networks. Our goal is to obtain a criterion whether a import network admits a circulation or not. The strategy will be to construct from an import network an s-t-network and to reduce the problem of the existence of a circulation in the import network to the existence of a maximal flow of a certain value in the derived s-t-network.

Given an import network $I = (\alpha : X \leftrightarrow X, \iota : X \leftrightarrow X, \omega : X \leftrightarrow X)$ we construct an s-t-network $N = (\hat{\alpha} : \hat{X} \leftrightarrow \hat{X}, s, t)$ as follows: $\hat{X}$ is given by $\hat{X} = X \dot{\cup} \{s\} \dot{\cup} \{t\}$. $\hat{\alpha}$ is defined by $\hat{\alpha} = \alpha_1 \sqcup \alpha_2 \sqcup \alpha_3$ with $\alpha_1 = \bigsqcup_{x \in X} |\iota x| s \lceil \nabla \rceil \lceil x \rceil$, $\alpha_2 = \lceil \alpha \rceil$ and $\alpha_3 = \bigsqcup_{x \in X} |\omega x| \lceil x \rceil \lceil \nabla \rceil t$.

Due to the lack of space we omit the proof that N is indeed an s-t-network and concentrate on the essential theorem of this paper:

Theorem 4.1. *There is a circulation on I iff the value of a maximal flow φ on N equals both $|\iota|$ and $|\omega|$.*

Proof. As a short preliminary we show that $c(s) = |\iota$ and $c(t^c) = |\omega|$ hold. Therefore we calculate $s\hat{\alpha}s^c = s(\alpha_1 \sqcup \alpha_2 \sqcup \alpha_3)s^c = s\alpha_1 s^c \sqcup s\alpha_2 s^c \sqcup s\alpha_3 s^c$. By rather simple considerations we can see that only the term $s\alpha_1 s^c$ does not become $\mathbf{0}_{\hat{X}\hat{X}}$, so we have $c(s) = |s\hat{\alpha}s^c| = |s\alpha_1 s^c| = |s \bigsqcup_{x \in X} |\iota x| s \lceil \nabla \rceil \lceil x \rceil|$. Because of $ss = s$, disjointness of the big join and distributivity we can replace this by $\Sigma_{x \in X} |\iota x|$, which is by similar reasons the same as $|\iota \bigsqcup_{x \in X} x|$, which yields $|\iota|$. Analogously we obtain $c(t^c) = |\omega|$. Furthermore we have by construction $\alpha_1 = s\alpha = s\alpha s^c$, $s\alpha_2 = s\alpha_3 = \mathbf{0}_{\hat{X}\hat{X}}$ and analogous properties for t.

Let now be φ a maximal flow on N with value $|\iota| = |\omega|$. Then we claim that $\lfloor \varphi \rfloor$ is a circulation on I. Obviously, due to the properties of embedding and projection, $\lfloor\varphi\rfloor$ respects the capacity constraint on I. Let now τ be an arbitrary test on X. Because of the flow conservation of φ we have $|\varphi\lceil\tau\rceil| = |\lceil\tau\rceil\varphi|$. From here on we can conclude as follows:

$$
\begin{aligned}
& |\varphi\lceil\tau\rceil| = |\lceil\tau\rceil\varphi| \Rightarrow \\
= \quad & \{\!\![\ id_{\hat{X}} = s\dot{\sqcup}X\dot{\sqcup}t\ ]\!\!\} \\
& |(s\dot{\sqcup}X\dot{\sqcup}t)\varphi\lceil\tau\rceil| = |\lceil\tau\rceil\varphi(s\dot{\sqcup}X\dot{\sqcup}t)| \Rightarrow \\
= \quad & \{\!\![\ \text{disjointness}, \varphi \sqsubseteq \hat{\alpha}, \text{construction of } \hat{\alpha}\ ]\!\!\} \\
& |s\varphi\lceil\tau\rceil| + |X\varphi\lceil\tau\rceil| = |\lceil\tau\rceil\varphi t| + |\lceil\tau\rceil\varphi X| \Rightarrow \\
= \quad & \{\!\![\ \varphi \text{ saturates } s \text{ and } t^c, \text{remarks above}\ ]\!\!\} \\
& |s\hat{\alpha}\lceil\tau\rceil| + |X\varphi\lceil\tau\rceil| = |\lceil\tau\rceil\hat{\alpha}t| + |\lceil\tau\rceil\varphi X| \Rightarrow \\
= \quad & \{\!\![\ \text{remarks above, construction of } \hat{\alpha}\ ]\!\!\} \\
& |\iota\tau| + |\lfloor\varphi\rfloor\tau| = |\tau\omega| + |\tau\lfloor\varphi\rfloor| \Rightarrow \\
= \quad & \{\!\![\ \lfloor\varphi\rfloor \sqsubseteq \alpha,\ \alpha, \omega\iota \sqsubseteq 0.5id_X, \text{properties of } \oplus\ ]\!\!\} \\
& |(\lceil\varphi\rceil \oplus \iota)\tau| = |\tau(\lceil\varphi\rceil \oplus \omega)|
\end{aligned}
$$

Let now γ be a circulation on I. Then we construct a fuzzy endorelation φ on $\hat{X}$ by $\varphi = \lceil\gamma\rceil \sqcup \bigsqcup_{x\in X} |\iota x| s\lceil\nabla\rceil\lceil x\rceil \sqcup \bigsqcup_{x\in X} |\omega x|\lceil x\rceil\lceil\nabla\rceil t$. By construction, φ satisfies the capacity constraint of N; the flow conservation can be showed similar as above. φ saturates both the cuts s and t^c. Because of $c(s) = |\iota|$ and $c(t^c) = \omega$ φ is indeed a maximal flow on N with value $|\iota| = |\omega|$.

5 Conclusion

We gave a short sketch of how to apply Kawahara's methods on other related network problems. The proofs remained naturally a little bit sketch-like, but it became visible how these methods can be extended for the use in other areas. As the mathematical structure of fuzzy relations is a Kleene algebra and automated reasoning in Kleene algebra is onthe rise (cf. [Höf]) we could await the first machine-generated proofs of theorems concerning networks.

References

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