Import Networks, Fuzzy Relations and Semirings

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About

- algebraic approach to networks
- using fuzzy relations
- Kleene algebra

Previous Work

- Yasuo Kawahara: On the Cardinality of Relations, 2006
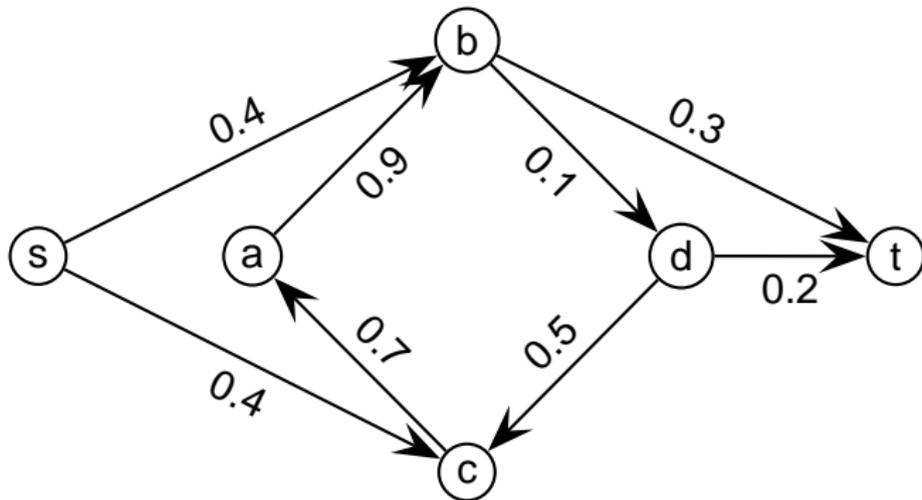- Roland Glück: Networks, Semirings and Fuzzy Relations, 2007

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Basic Definitions
Operations on Fuzzy Relations
Test Relations
Cardinality of Fuzzy Relations

Basics

Definition

A *fuzzy relation* $\alpha : X \leftrightarrow Y$ between a set X and a set Y is a function $\alpha : X \times Y \to [0, 1]$

- natural generalisation of traditional relations
- total function!
- notation $Rel(X, Y)$ for set of all fuzzy relations between X and Y

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Visualisation of Fuzzy Relations

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Special Fuzzy Relations

- *empty relation* $0_{XY} : X \leftrightarrow Y$ with $0_{XY}(x, y) = 0$
- *universal relation* $\nabla_{XY} : X \leftrightarrow Y$ with $\nabla_{XY}(x, y) = 1$
- *identity relation* $id_X : X \leftrightarrow X$ with $id_X(x, y) = \delta_{xy}$

Operations on Fuzzy Relations

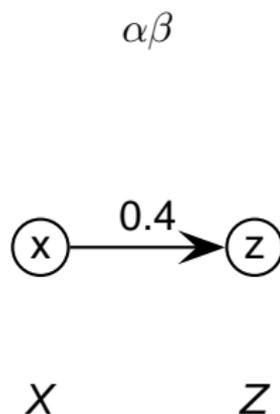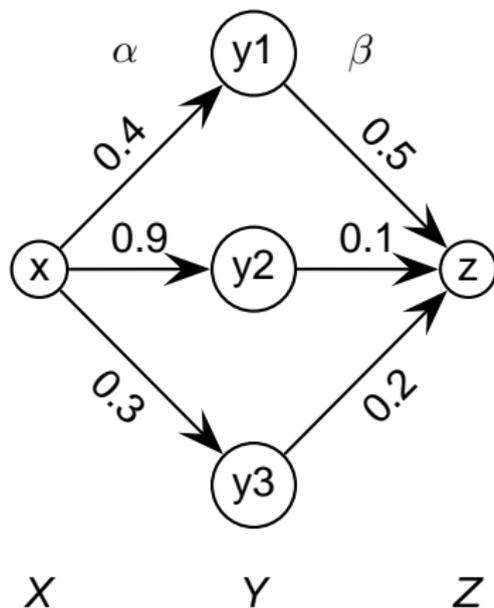Let $\alpha : X \leftrightarrow Y$, $\beta : X \leftrightarrow Y$ be fuzzy relations. Then we define:

- the *join* $\alpha \sqcup \beta : X \leftrightarrow Y$ by
  $(\alpha \sqcup \beta)(x, y) := max\{\alpha(x, y), \beta(x, y)\}$
- the *meet* $\alpha \sqcap \beta : X \leftrightarrow Y$ by
  $(\alpha \sqcap \beta)(x, y) := min\{\alpha(x, y), \beta(x, y)\}$
- the *converse* $\alpha^\sharp : Y \leftrightarrow X$ by
  $\alpha^\sharp(y, x) := \alpha(x, y)$
- the *scalar multiplication* $k \cdot \alpha$ for $k \in [0, 1]$ by
  $k \cdot \alpha(x, y) := k \cdot \alpha(x, y)$

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Composition of Fuzzy Relations

Let $\alpha : X \leftrightarrow Y$, $\beta : Y \leftrightarrow Z$ be fuzzy relations. Then we define:

- the *composition* $\alpha \circ \beta : X \leftrightarrow Z$ by
  $\alpha \circ \beta(x, z) = max_{y \in Y} \, min\{\alpha(x, y), \beta(y, z)\}$

- abbreviation $\alpha\beta$ for $\alpha \circ \beta$

- maximally possible amount from x to z via an element of Y

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Example Composition

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Ordering

Lemma

The relation $\sqsubseteq$ on Rel(X, Y), defined by

$$\alpha \sqsubseteq \beta \Leftrightarrow \alpha(x, y) \leq \beta(x, y) \ \forall (x, y) \in X \times Y$$

is a partial order on Rel(X, Y).

- Note that $\alpha \sqsubseteq \beta \Leftrightarrow \alpha \sqcup \beta = \beta$

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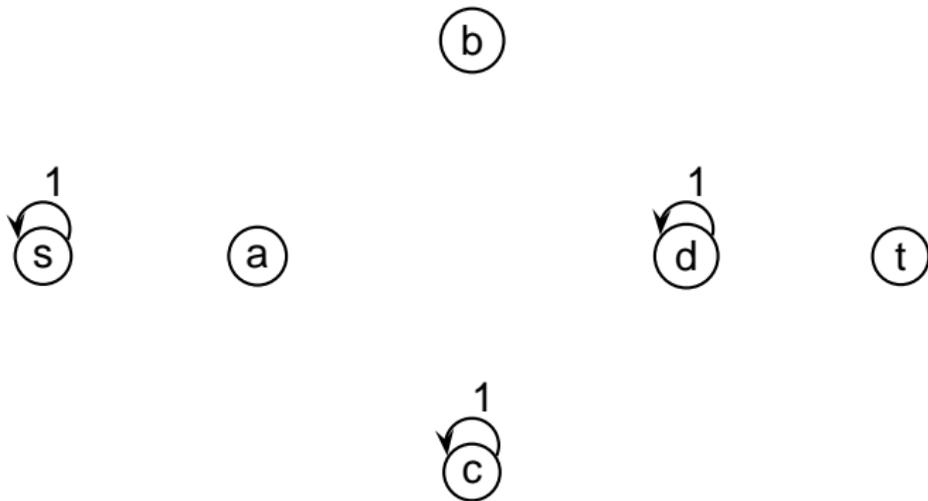
Surprise, Surprise!

Theorem

The fuzzy endorelations over an arbitrary set X form an idempotent semiring with join as addition, composition as multiplication, 0_{XX} as zero and id_X as one. Its natural order is $\sqsubseteq$. and it can be extended to a Kleene algebra.

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Test Relations

- A *test relation* on X is a subrelation of the identity relation id_X with a range contained in $\{0, 1\}$.
- 1:1-correspondence between test relations on X and subsets of X
- A *point relation* is a test relation corresponding to a singleton subset.
- They can be used to restrict fuzzy relations to edges beginning or ending in subsets of X.

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Example Test Relation

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Definition of Cardinality

Definition

The *cardinality* $|\alpha|$ of a fuzzy relation $\alpha : X \leftrightarrow Y$ is defined by

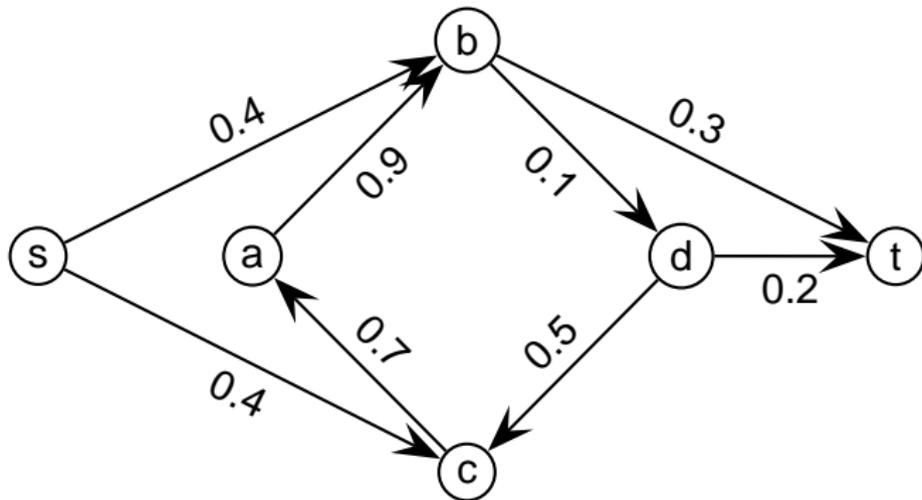$$|\alpha| = \sum_{(x,y) \in X \times Y} \alpha(x, y)$$

- $|\cdot|$ can be ∞ for infinite sets X or Y
- from now on only fuzzy relations on finite sets

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Definition

Definition (Kawahara 2006)

An *s-t-network N* is a triple $N = (\alpha : X \leftrightarrow X, s, t)$, where α is a fuzzy endorelation on X with $\alpha \sqcap \alpha^\sharp = 0_{XX}$ and s (the *source*) and t (the *sink*) are two distinct elements of X.

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Example s-t-network

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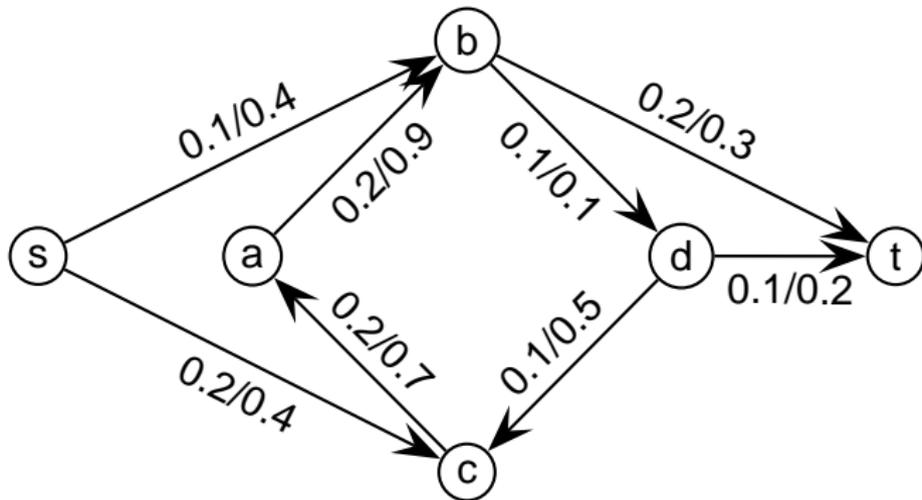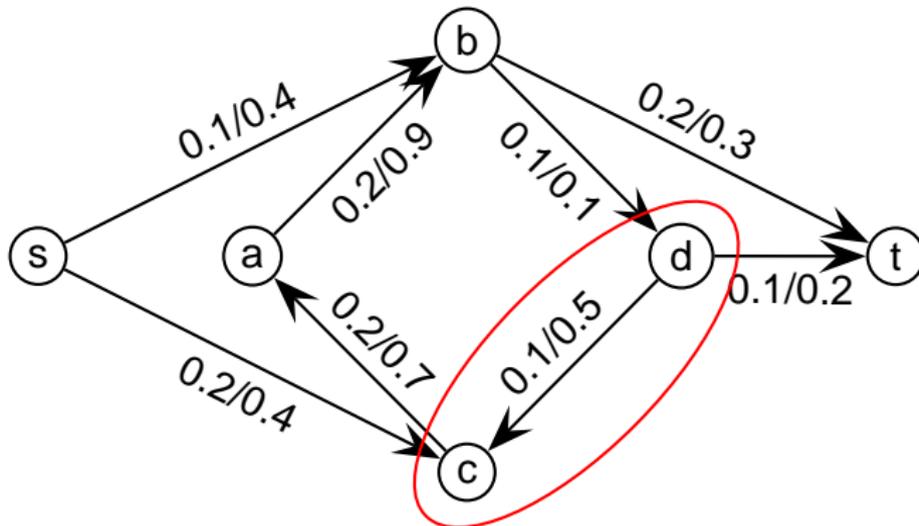Circulations

Definition

Definition

A *flow* φ on an s-t-network $N = (\alpha : X \leftrightarrow X, s, t)$ is a fuzzy endorelation on X with the properties

- $\varphi \sqsubseteq \alpha$ (capacity constraint)

- $|\tau\varphi| = |\varphi\tau|$ for all test relations τ on X with $\tau \sqcap (s \sqcup t) = 0_{XX}$ (flow conservation)
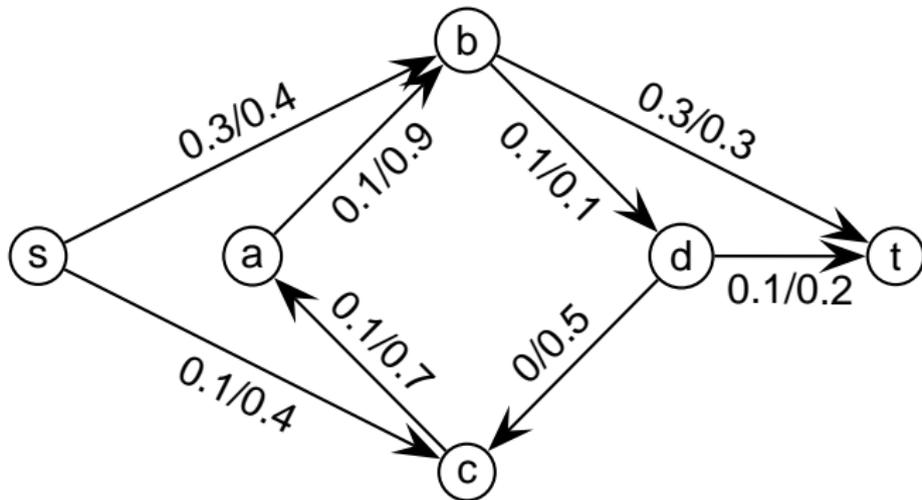
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Example Flow

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Example Flow

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Value and Cut

- The *value* *val*(φ) of a flow φ is defined as
  $val(\varphi) = |s\varphi| - |\varphi s|$
- A flow is called *maximal*, if its value is maximal

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Maximal Flow

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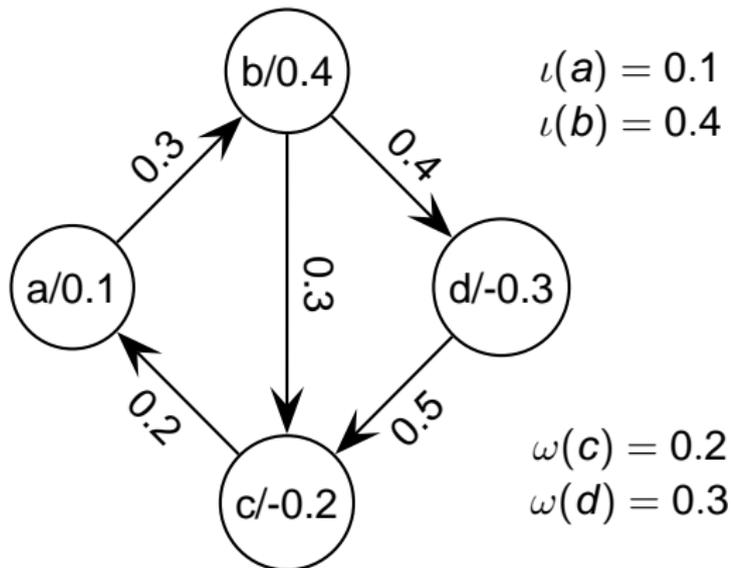Circulations

Definition

Definition

An *import network* I is a triple $I = (\alpha : X \leftrightarrow X, \iota, \omega)$ with the properties

- $\alpha \sqsubseteq 0.5 \cdot \nabla_{XX}$
- $\alpha \sqcap \alpha^\sharp = 0_{XX}$
- $\iota, \omega \sqsubseteq 0.5 \cdot id_X$
- $\iota \sqcap \omega = 0_{XX}$

α is called the *capacity function*, ι the *import* and ω the *outflow*.

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Visualisation of Import Networks

$\iota(a) = 0.1$
$\iota(b) = 0.4$

$\omega(c) = 0.2$
$\omega(d) = 0.3$

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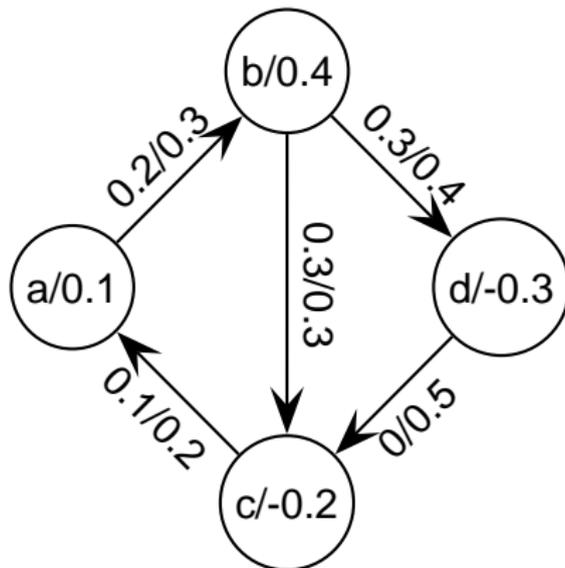## Circulations on Import Networks

Definition

A *Circulation* φ on an import network $I = (\alpha : X \leftrightarrow X, \iota, \omega)$ is a fuzzy endorelation on X with the properties
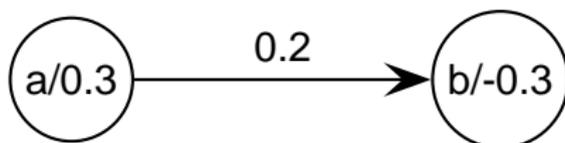
- $\varphi \sqsubseteq \alpha$ (capacity constraint)
- $|(\varphi \oplus \iota)\tau| = |\tau(\varphi \oplus \omega)|$ (flow conservation)

where $(\alpha \oplus \beta)(x, y) := min\{1, \alpha(x, y) + \beta(x, y)\}$.

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Example Circulation

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Import Network without Circulation

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Existence and Determining of Circulations

- Existence of a circulation
- Determining a circulation

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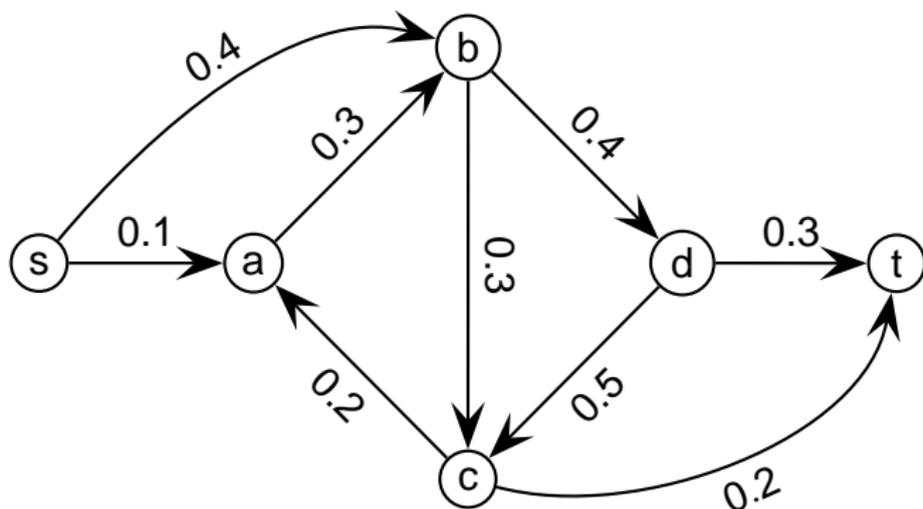## Existence and Determining of Circulations

- Existence of a circulation
- Determining a circulation
- Both via an s-t-network

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Example Import Network

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Associated s-t-Network

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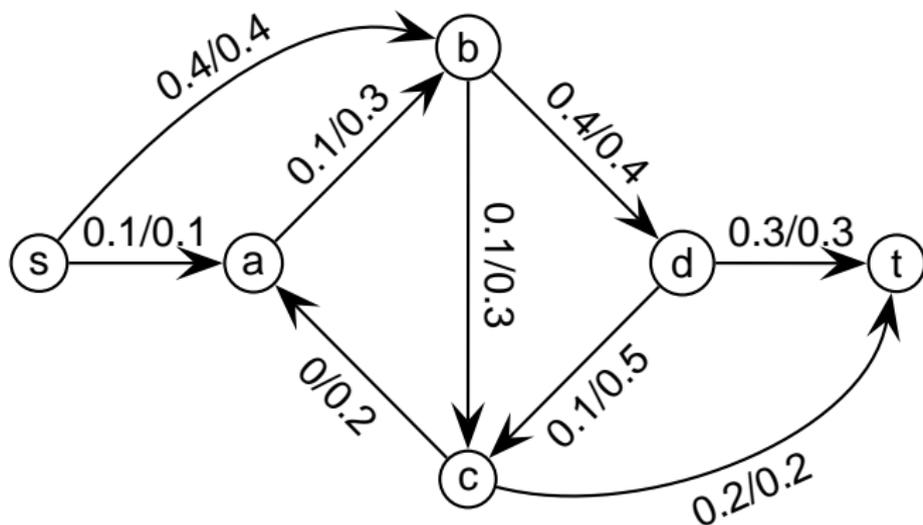Import Networks
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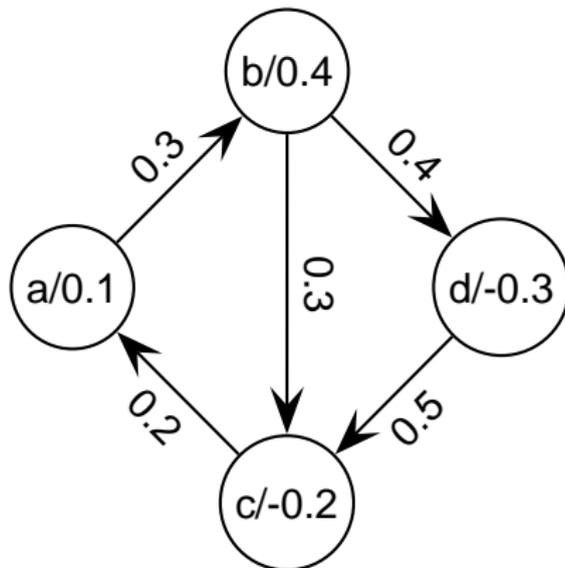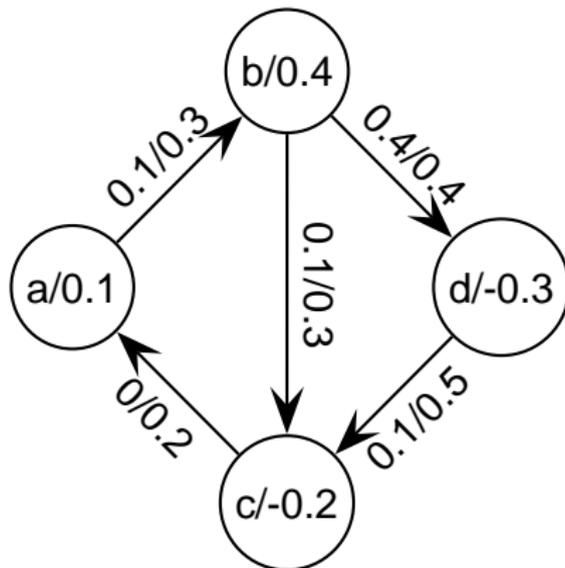
Existence Criterion for Circulations

Theorem

There is a circulation on an import network iff the value of a maximal flow in the associated s-t-network equals both $|\iota|$ and $|\omega|$.

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Maximal Flow in the Associated s-t-Network

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Resulting Circulation

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Summary

We saw

- an algebraic description of network problems
- application Kleene algebra in action
- first steps towards automated proving of theorems concerning networks

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To Do's

- feeding automated provers with this stuff
- applying similar methods to related problems (partially done, MPC08)
- describing graph theoretical problems with these methods

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Enjoy Your Meal!

Questions?
Criticisms?
Suggestions?