On the Computation of Isolated Sublattices

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Abstract. In this short notice we give some ideas how to compute isolated sublattices which can be used to derive a recursive algorithm for the computation of the number of closure operators on a finite lattice. We give an asymptoticaly optimal algorithm for deciding the existence and - in the case of existence - the computation of useful nontrivial isolated summit sublattices. The general case (i.e., an optimal algorithm for the computation of general nontrivial useful isolated sublattices) remains unsolved, however, we try to give some ideas and hints for future research.

1 Motivation, Basic Definitions and Notation

Isolated Sublattices are used in [1] to derive a recursive algorithm for computing thee number of closure operators by means of a quotient lattice. A crucial point in this algorithm is the computation of an isolated sublattice. This can be done in polynomial time as sketched at the end of [1], however, it would be desirable to obtain a linear time algorithm. We will see in Subsection 2.1 that we can achieve this for a special class of isolated sublattices. For the general case, we sketch some ideas in Subsection 2.2.

Let us start with some definition and notational conventions.

For a lattice $(S, \leq)$, we denote its least and greatest element by $\bot$ and $\top$, resp. We often use S as notation for a lattice instead of $(S, \leq)$. In a lattice, we define an *isolated sublattice* as follows:

Definition 1.1. *Let* $(S, \leq)$ *be a lattice. A subset* $S' \subseteq S$ *is called an* isolated sublattice *if it fulfills the following properties:*

1. S' *is a sublattice with greatest element* $\top_{S'}$ *and least element* $\bot_{S'}$.
2. $\forall x \notin S' \forall y' \in S' : y' \leq x \Rightarrow \top_{S'} \leq x$
3. $\forall x \notin S' \forall y' \in S' : x \leq y' \Rightarrow x \leq \bot_{S'}$

Clearly S is an isolated sublattice which we call *trivial*. Also, every singleton subset of S is also an isolated sublattice, even though of little interest and use. So we call an isolated sublattice with more than one element *useful*.

In [1] it is shown that an isolated sublattice always has the form $[\bot_{S'}, \top_{S'}]$, i.e., it is an interval in S. If $\top_{S'} = \top$ holds we speak of a *summit isolated sublattice*. In particular, every summit isolated sublattice is of the form $[z, \top]$. Moreover, it is easy to check that both $[\bot, \top]$ and $[\top, \top]$ are summit isolated sublattices.

However, they are of no interest here because they do not lead to progress in the algorithm from [1].

In order to make use of isolated sublattices we need an algorithm for computing isolated sublattices in an finite lattice. Therefore, we assume that the lattice is given by its Hasse diagram as a directed graph $G = (S, E)$ where the edges point "upwards". i.e., $(u, v) \in E$ implies $u < v$. For the reverse graph of $G = (S, E)$ we use the notation $G^{\leftarrow} = (S, E^{\leftarrow})$ and for its undirected version we write $G^{\leftrightarrow}$., Given a path $p = v_1 v_2 \ldots v_n$ we write $v \in p$ if $v = v_i$ for some $i \in [1, n]$ and say that *p contains v*. We call a node u a (v, w)-separator if every path from v to w contains u.

2 Connection with Separators

2.1 Summit Isolated Sublattices and Separators

First, we give an alternative characterization than Definition 1.1 that $[z, \top]$ is a summit isolated sublattice. To this end, we observe that $[z, \top]$ is a lattice and hence fulfills already the first condition of Definition 1.1. Next, we can omit the second condition of this definition since the antecedent $y' \leq x$ implies $x \in [z, \top]$ due to $x \leq \top$, contradicting the assupumtion $x \notin S'$ from the definition. So we have the following lemma: ,

Lemma 2.1. $[z, \top]$ *is a summit isolated sublattice iff the following implication holds:*

- $\forall x \notin [z, \top] \forall y' \in [z, \top] : x \leq y' \Rightarrow x \leq z.$

Using this characterization, we can give a still simpler one:

Lemma 2.2. $[z, \top]$ *is a summit isolated sublattice iff for all* $x \notin [z, \top]$ *the inequality* $x \leq z$ *holds.*

Proof: "⇒": Let us pick an arbitrary $x \notin [z, \top]$. Clearly, we have $x \leq \top$, and the claim follows from the substitution $y' := \top$ in Lemma 2.2.
"⇐": Since $x \leq z$ holds by assumption for all $x \notin [z, \top]$ the consequent from Lemma 2.2 is always true. ■

After this characterization which is valid in arbitrary lattices we can give a first version for finite lattices:

Lemma 2.3. *Let* $(S, \leq)$ *be a finite lattice with Hasse diagram* $G = (S, E)$*. Then every summit isolated sublattice of S has the form* $[z, \top]$ *where z is a* $(\top, \bot)$*-separator in* $G^{\leftarrow}$*.*

Proof: As already mentioned in Section 1 every summit isolated sublattice has the form $[z, \top]$ so it remains to show the condition concerning the separation property of z. We omit thee trivial and uninteresting cases $z = \bot$ and $z = \top$ and consider a path $p = v_1 v_2 \ldots v_n$ in $G^{\leftarrow}$ with $v_1 = \top$ and $v_n = \bot$. Then there is an index $i \in [1, n - 1]$ with $v_i \in [z, \top]$ and $v_{i+1} \notin [z, \top]$ and let us assume for the sake of contradiction that $v_i \neq z$ holds. Due to $v_i \in [z, \top]$ this implies $z < v_i$ and the property $(v_i, v_{i+1}) \in E^{\leftarrow}$ implies $v_{i+1} < v_i$. On the other hand, Lemma 2.2

implies $v_{i+1} < z$. Alltogether, we have $z < v_i$ and $v_{i+1} < z$. But then $v_{i+1} < v_i$ follows already by transitivity of $<$, hence (v_{i+1}, v_i) can not be an edge in the Hasse diagram of S and now $(v_i, v_{i+1}) \notin E^{\leftarrow}$ contradicts the path property of p. ■

Problems like separators in directed graphs can be tackled by modified max-flow algorithms which are in general somehow cumbersome to implement. Luckily, there is a characterization using undirected graphs:

Theorem 2.4. *Let $(S, \leq)$ be a finite lattice with Hasse diagram $G = (V, E)$. Then every summit isolated sublattice of S has the form $[z, \top]$ where z is a $(\top, \bot)$-separator in $G^{\leftrightarrow}$.*

Proof: The claim is obvious for the case $z \in \{\bot, \top\}$ so let us assume that $z \notin \{\bot, \top\}$ holds and let us fix an arbitrary path $p = v_1 v_2 \ldots v_n$ in $G^{\leftrightarrow}$ with $v_1 = \top$ and $v_n = \bot$. Analogously to the proof of Lemma 2.3 there are vertices v_i and v_{i+1} with $v_i \in [z, \top]$ and $v_{i+1} \notin [z, \top]$, and here, too, we claim that $v_i = z$ holds. Due to $v_i \in [z, \top]$ and $v_{i+1} \notin [z, \top]$ we have $(v_i, v_{i+1}) \in E^{\leftarrow}$. On the other hand, from $v_i \in [z, \top]$ we conclude that there is a path $p' = v'_1 v'_2 \ldots v'_{n'}$ in $G^{\leftarrow}$ with $v'_1 = \top$, $v'_{n'} = v_i$ and $v'_{i'} \neq z$ for all $i' \in [1, n']$. Analogously, there is a path $p'' = v''_1 v''_2 \ldots v''_{n''}$ in $G^{\leftarrow}$ with $v''_1 = v_{i+1}$, $v''_{n''} = \bot$ and $v''_{i''} \neq z$ for all $i'' \in [1, n'']$. This means that $v'_1 v'_2 \ldots v'_{n'} v''_1 v''_2 \ldots v''_{n''}$ is a path in $G^{\leftarrow}$ from $\top$ to $\bot$, and the claim follows from Lemma 2.3. ■

Using a result from [2] this leads to the following corollary:

Corollary 2.5. *Given the Hasse diagram (S, E) of a finite lattice S, it can be determined in $O(|E|)$ time whether S has a nontrivial useful summit isolated sublattice. In the case of existence, a nontrivial useful summit isolated sublattice can be determined also in $O(|E|)$ time.*

Clearly, this time bound is asymptoticaly optimal. Note that [2] uses only a simple DFS and does not rely on some sophisticated network flow algorithms. Moreover, we can determine (in the case of existence) an inclusion-maximal nontrivial useful summit isolated sublattice in time linear in $|E|$.

2.2 General Isolated Sublattices and Separators

An approach similar to above leads to the following lemma in the general case:

Lemma 2.6. *Let $(S, \leq)$ be a finite lattice with Hasse diagram $G = (S, E)$. Then all isolated sublattices of S are exactly the intervals $[x, y]$ where x is a $(y, \bot)$-separator in $G^{\leftarrow}$ and y is an $(x, \top)$-separator in G.*

Compared to the computation of summit isolated sublattices we face a much more adversary situation if we want to compute general isolated sublattices. First, we now nothing about the top element of such an isolated sublattices (in a summit isolated sublattice the top element is always the greatest element of the lattice itself). Second, there may be a superlinear number of general sublattices: A chain of length n has $\frac{n(n-1)}{2} - 1$ nontrivial useful isolated sublattices (a linear amount of them, namely $n - 1$, are summit isolated sublattices). This seems to make it impossible to come up with a linear time algorithm as in the case of summit isolated sublattices. However, we are not interested in all isolated sublattices but only in inclusion-maximal ones. Since distinct inclusion-maximal isolated sublattices are disjoint (see [1]) there at most $|S|$ of them. Moreover, we even do not need to know all inclusion-maximal isolated sublattices but are satisfied with some inclusion-maximal useful nontrivial isolated sublattice. This makes hope for a linear time algorithm, however, some clever ideas seem to be necessary. In the sequel we list some observations we may help in this direction.

Transitivity of the Separator Property. If v_2 is a (v_1, v)-separator and v_3 is a (v_2, v)-separator then v_3 is a (v_1, v)-separator. This reflects the fact that for isolated sublattices $[v, v_1]$ and $[v, v_2]$ the greatest elements v_1 and v_2 are comparable (see [1]).

Supremum/Infimum Property of Lattices. The ideas presented so far do not fully exploit the properties of a lattice but rely only on those of a finite order possessing a least and a greatest element. Is it possible to take advantage of the existence of suprema/infima, e.g., that for pairwaise distinct v_1, v_2, v_3 and v_4 the existence of the edges (v_1, v_3) and (v_2, v_3) excludes the existence of the edge (v_1, v_4) (and similar properties for paths)?

3 Summary and Discussion

This fragmentary note shows that summit isolated sublattices can be computed in linear but leaves the question open for the general case. We hope that this stimulates further research towards the computation of isolated sublattices.

References

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