Power and Limits of Model Refinement

Roland Glück

Universität Augsburg

19.07.2010
Sion

Overview

- recent work
- dioids and models
- refinement
- conclusion and outlook

Use of Quotient Models

- work by Sintzoff, Möller, and Glück
- control and refinment via quotients
- Step 1: shrinking a model by bisimulation quotient
- Step 2: refining the shrunk model
- Step 3: expanding the refined model

Crucial Point

- step 1 and 3 are easily treatable
- step 2 is a crucial point
- question: which properties can be ensured via refinement?
- here: exploration of optimality properties

Definition of Dioids

Definition

A *dioid* D is a quintuple $(S, \sqsubseteq, \cdot, 1, 0)$ where (S, ord) is a complete lattice with linear order $\sqsubseteq$ and least element 0, and $\cdot$ is a associative mapping $\cdot : S \times S \to S$ with neutral element 1 which distributes over arbitrary suprema wrt. $\sqsubseteq$.

The binary supremum operator is written as $+$, the general as $\sum$. $\cdot$ is often omitted.

Using Dioids

- example dioids:
  - $(\mathbb{R} \cup \pm\infty, \geq, \infty, +, 0)$
  - $(\mathbb{R} \cup \pm\infty, \leq, -\infty, min, \infty)$
- used as edge labels in graphs

Definition of Models

Definition

A *model* M is a structure $((V,E),(S,\sqsubseteq,\cdot,1,0),c)$ where (V,E) is a graph with finite node set V and edge set E, $(S,\sqsubseteq,\cdot,1,0)$ is a dioid and $c : E \rightarrow S$ is the *cost function* of M. (V,E) is called its *associated graph*, $(S,\sqsubseteq,\cdot,1,0)$ its *associated dioid*.

Costs in Models

Definition

For a model $M = ((V, E), (S, \sqsubseteq, \cdot, 1, 0), w)$ and a walk $w = x_1 x_2 \dots x_n$ the *cost* $c(w)$ of w in M is defined as $c(w) = c(x_1, x_2) \cdot c(x_2, x_3) \dots c(x_{n-1}, x_n)$. The distance $d(x, y)$ of two nodes $x, y \in E$ in M is defined as $d(x, y) = \sum_{w \in W_M(x,y)} c(w)$, where $W_M(x, y)$ denotes the set of all walks from x to y in (V, E).

Remarks on Costs

- natural generalisation of shortest walk, maximum capacity, highest reliability and related problems
- costs of a walk and distance of two nodes do always exist
- no formalisation of longest path

Optimal Walks

Definition

A walk $w = x_1 x_2 \ldots x_n$ in a model M is called *optimal* if $c(w) = d(x_1, x_n)$

- optimal walks do not necessarily exist
- criterion(a) for existence

Selective Dioids

Definition

A dioid is called *selective* if its 1 is the greatest element wrt. $\sqsubseteq$.

- equivalent to $a \sqsubseteq b \Rightarrow ac \sqsubseteq b$ and $a \sqsubseteq b \Rightarrow ca \sqsubseteq b$
- natural requirement
- $(\mathbb{R}_0^+ \cup \infty, \geq, \infty, +, 0)$ is selective
- $(\mathbb{R} \cup \pm\infty, \leq, -\infty, +, 0)$ is not selective

Consequences of Selectivity

- $abc \sqsubseteq ac$
- prolonging a walk does not make the walk better
- sufficient criterion for existence of optimal paths

Selectivity implies Optimal paths

Theorem

In a model with selective associated dioid there is always an optimal walk between two nodes, and there is even an optimal path between two nodes.

Proof.

Idea: For every walk w between two nodes x and y there is due to selectivity a path p between x and y with $c(w) \sqsubseteq c(p)$. □

Target Model and Submodels

Definition

A model $M' = ((V', E'), (S, \sqsubseteq, \cdot, 1, 0), w')$ is called a *submodel* of a model $M = ((V, E), (S, \sqsubseteq, \cdot, 1, 0), w)$ if $(V', E') \subseteq (V, E)$ and $w' = w|_{E'}$. A structure $M = ((V, E), (S, \sqsubseteq, \cdot, 1, 0), w, v)$ where $M = ((V, E), (S, \sqsubseteq, \cdot, 1, 0), w)$ is a model and $v \in V$ is called a *target model* with target v. Target submodels are defined in obvious manner.

Explanation

- submodels formalise refinement by removing edges and nodes
- target models for optimal paths to distinguished nodes
- cf. all routes from towns in Germany leading to Munich

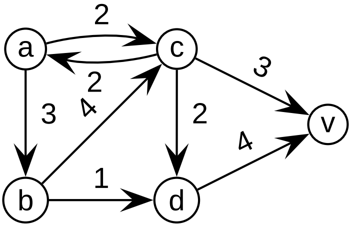
Optimal Submodels

Definition

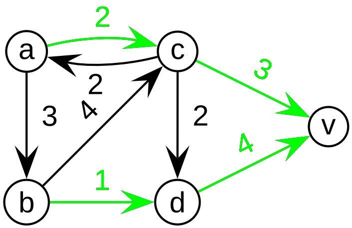
A target submodel M' of a target model with target v is called *optimal* if for all walks w in M' leading to from an arbitrary node x into v the cost of w equals the distance between x and v in M and v is reachable from every node in M'.

Roads marked by direction signs to Munich should determine an optimal submodel ...

Examples for Optimal Submodels

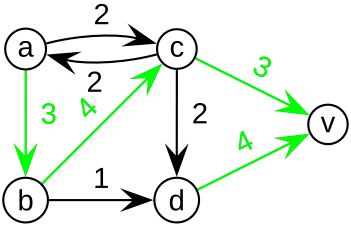


Examples for Optimal Submodels



As shortest walk problem

Examples for Optimal Submodels



As maximum capacity problem

Refineability

- optimal submodels need not to exist
- optimal submodels need not to be unique
- mainly question of existence and computability

Definition

A target model is called *refineable*, if it has an optimal submodel.

Sufficient and Necessary Criterion for Refineability

Theorem

Let $\mathcal{M}$ be the set of all target models with common associated diod D. Then every target model $M \in \mathcal{M}$ is refineable iff D is selective.

Proof

$\Leftarrow$:

Consider the following Dijkstra-like algorithm (after Gondran-Minoux):

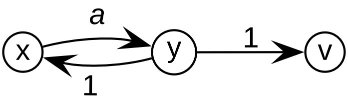
$d(v) = 1;$
$d(x) = 0 \; \forall x \in V \backslash \{v\};$
$T = V;$
while $T \neq \emptyset$
choose $x \in T$ with $d(x) = \sum_{y \in T} d(y);$
$T = T \backslash \{x\};$
forall $y \in T$
$d(y) = d(y) + (d(x) \cdot c(y, x))$
end forall
end while

Proof

- algorithm determines $d(x, v)$ correctly
- in line 8 remember predecessor of y if $d(y)$ is updated
- construct submodel using predecessor lists
- does not deliver unique submodel
- e.g. complete graph with edge weights all equal to 1
- polynomial runtime

Proof

$\Rightarrow$:
Consider the following target model:



- refinement preserves (y, x): $1 = c(yv) = c(yxyv) = a$✓
- refinement removes (y, x): $a = c(yxyv) \sqsubseteq d(y, v) = 1$✓

Conclusion

- refinement in important cases possible
- quadratic runtime
- complete solution for bisimulation algorithm

Future Work

- extension to set valued target models
- treatment of non-selective diods
- study of other control goals (temporal logic, termination, ...)

We hope you have

enjoyed the show