Circulations, Fuzzy Relations and Semirings

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Luminy

About

- algebraic approach to networks
- using fuzzy relations
- towards automated reasoning

Previous Work

- Yasuo Kawahara: On the Cardinality of Relations, 2006
- Roland Glück: Networks, Semirings and Fuzzy Relations, 2007

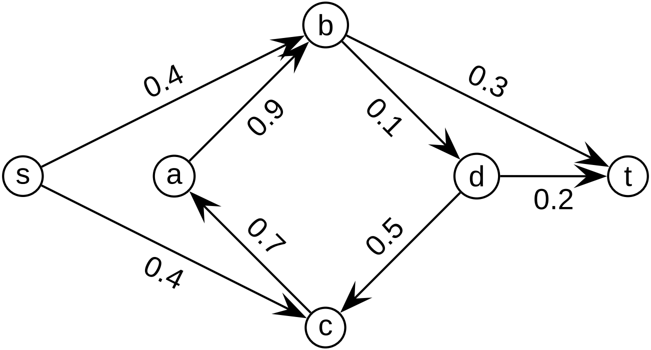
Basics

Definition

A *fuzzy relation* $\alpha : X \leftrightarrow Y$ between a set X and a set Y is a function $\alpha : X \times Y \to [0, 1]$

- natural generalisation of traditional relations
- total function!
- fuzzy relations between X and itself are called *fuzzy endorelations*
- edge weights in weighted graphs can be scaled into $[0, 1]$

Visualisation of Fuzzy Relations



Special Fuzzy Relations

- *empty relation* $0_{XY} : X \leftrightarrow Y$ with $0_{XY}(x,y) = 0$
- *universal relation* $\nabla_{XY} : X \leftrightarrow Y$ with $\nabla_{XY}(x,y) = 1$
- *identity relation* $id_X : X \leftrightarrow X$ with $id_X(x,y) = \delta_{xy}$

Operations on Fuzzy Relations

Let $\alpha : X \leftrightarrow Y$, $\beta : X \leftrightarrow Y$ be fuzzy relations. Then we define:

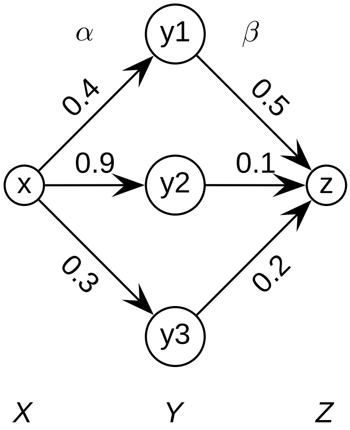
- the *join* $\alpha \sqcup \beta : X \leftrightarrow Y$ by
  $(\alpha \sqcup \beta)(x,y) := max\{\alpha(x,y), \beta(x,y)\}$
- the *meet* $\alpha \sqcap \beta : X \leftrightarrow Y$ by
  $(\alpha \sqcap \beta)(x,y) := min\{\alpha(x,y), \beta(x,y)\}$
- the *converse* $\alpha^\sharp : Y \leftrightarrow X$ by
  $\alpha^\sharp(y,x) := \alpha(x,y)$

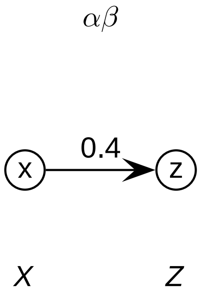
Composition of Fuzzy Relations

Let $\alpha : X \leftrightarrow Y$, $\beta : Y \leftrightarrow Z$ be fuzzy relations. Then we define:

- the *composition* $\alpha \circ \beta : X \leftrightarrow Z$ by
  $\alpha \circ \beta(x, z) = max_{y \in Y}\ min\{\alpha(x, y), \beta(y, z)\}$
- abbreviation $\alpha\beta$ for $\alpha \circ \beta$
- maximally possible amount from x to z via an element of Y

Example Composition





Ordering

Lemma

The relation

$$\alpha \sqsubseteq \beta \Leftrightarrow \forall (x,y) \in X \times Y : \alpha(x,y) \leq \beta(x,y)$$

is a partial order.

- Note that $\alpha \sqsubseteq \beta \Leftrightarrow \alpha \sqcup \beta = \beta$

Mathematical Structure

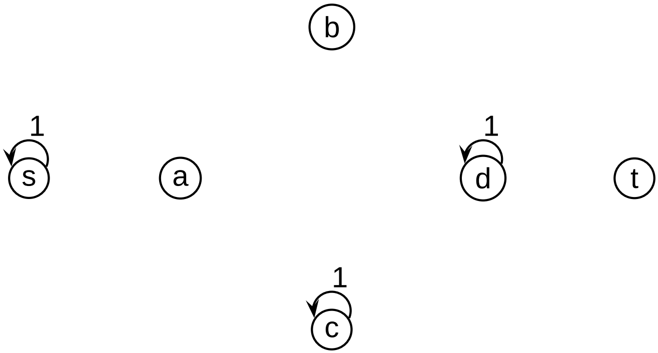
Theorem

The fuzzy endorelations over an arbitrary set X form an idempotent semiring with join as addition, composition as multiplication, 0_{XX} as zero and id_X as one. Its natural order is $\sqsubseteq$.

Test Relations

- A *test relation* on X is a subrelation of the identity relation id_X with a range contained in $\{0, 1\}$.
- 1:1-correspondence between test relations on X and subsets of X
- A *point relation* is a test relation corresponding to a singleton subset.
- Test relations can be used to restrict fuzzy relations to edges beginning or ending in subsets of X.

Example Test Relation



Definition of Cardinality

Definition

The *cardinality* $|\alpha|$ of a fuzzy relation $\alpha : X \leftrightarrow Y$ is defined by

$$|\alpha| = \sum_{(x,y)\in X\times Y} \alpha(x,y)$$

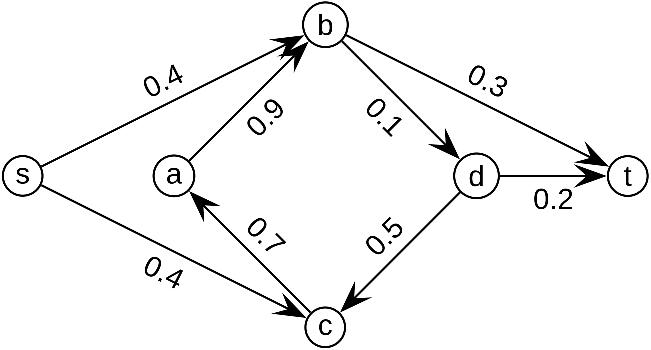
- $|\cdot|$ can be ∞ for infinite sets X or Y
- from now on only fuzzy relations on finite sets

Definition

Definition (Kawahara 2006)

An *s-t-network N* is a triple $N = (\alpha : X \leftrightarrow X, s, t)$, where α is a fuzzy endorelation on X with $\alpha \sqcap \alpha^\sharp = 0_{XX}$ and s (the *source*) and t (the *sink*) are two distinct elements of X.

Example s-t-network



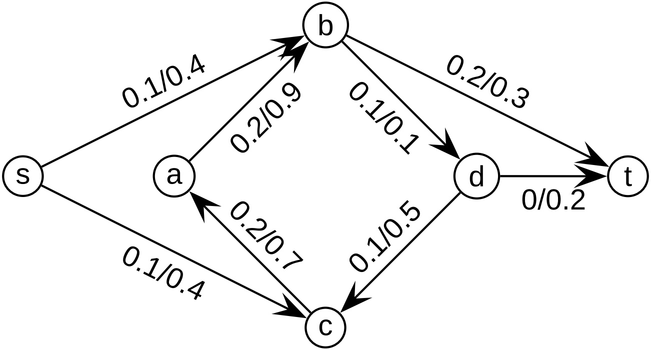
Definition

Definition

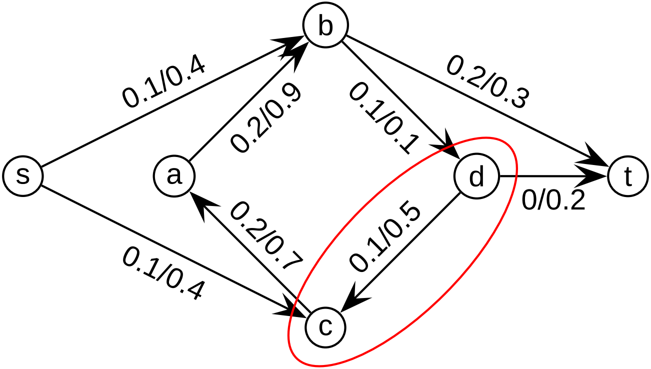
A *flow* φ on an s-t-network $N = (\alpha : X \leftrightarrow X, s, t)$ is a fuzzy endorelation on X with the properties

- $\varphi \sqsubseteq \alpha$ (capacity constraint)
- $|\tau\varphi| = |\varphi\tau|$ for all test relations τ on X with $\tau \sqcap (s \sqcup t) = 0_{XX}$ (flow conservation)

Example Flow



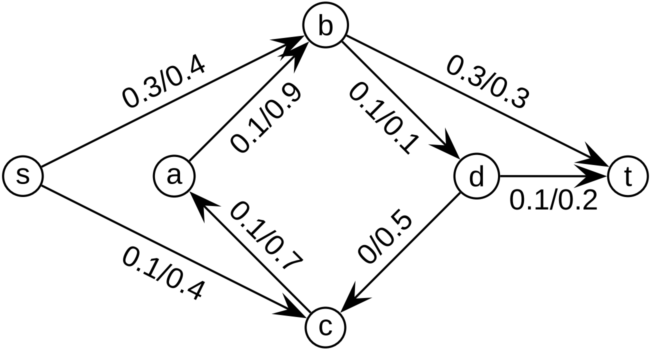
Example Flow



Value and Maximality of Flows

- The *value* $val(\varphi)$ of a flow φ is defined as $val(\varphi) = |s\varphi| - |\varphi s|$
- A flow is called *maximal*, if its value is maximal

Maximal Flow

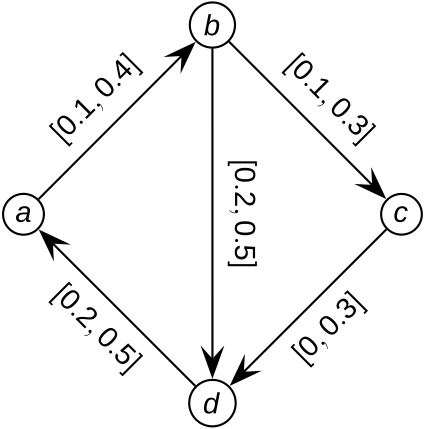


Definition

Definition

A *network with lower bounds* N is a tuple
$I = (\alpha : X \leftrightarrow X, \beta : X \leftrightarrow X)$ with the properties $\beta \sqsubseteq \alpha$ and
$\alpha \sqcap \alpha^\sharp = 0_{XX}$.
α and β are called the *upper bound* and the *lower bound*.

Visualisation of Networks with Lower Bounds



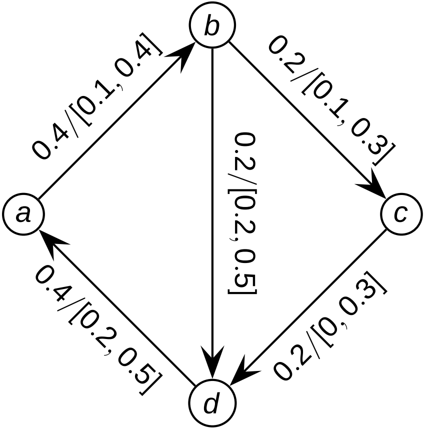
Circulations in Networks with Lower Bounds

Definition

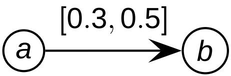
A *Circulation* φ in a network with lower bounds $N = (\alpha : X \leftrightarrow X, \beta : X \leftrightarrow X)$ is a fuzzy endorelation on X with the properties

- $\beta \sqsubseteq \varphi \sqsubseteq \alpha$ (capacity constraint)
- $|\varphi\tau| = |\tau\varphi|$ for all test relations τ (flow conservation)

Example Circulation



Network With Lower Bounds without Circulation



Existence and Determining of Circulations

- Existence of a circulation
- Determining a circulation

Existence and Determining of Circulations

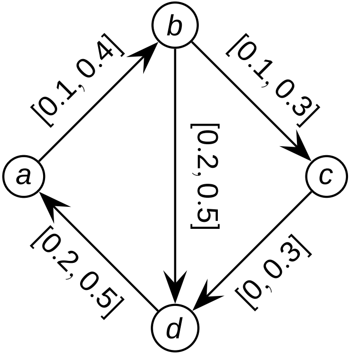
- Existence of a circulation
- Determining a circulation
- Both via an s-t-network

Construction of an associated s-t-Network

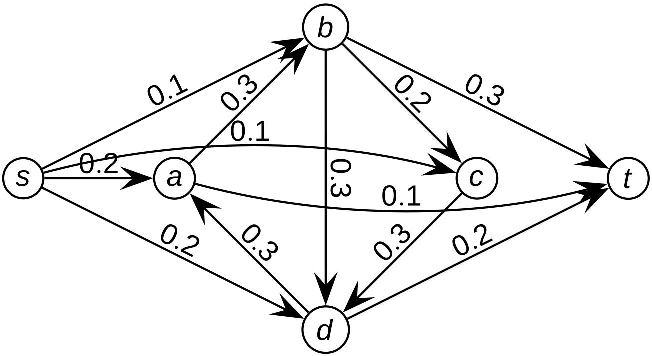
From a network with lower bounds $N = (\alpha : X \leftrightarrow X, \beta : X \leftrightarrow X)$ we construct the associated s-t-network $N' = (\alpha' : X' \leftrightarrow X', s, t)$ as follows:

- set $X' := X \dot{\cup} \{s, t\}$
- set $\alpha'(s, x) := \Sigma_{y \in X} \beta(y, x)$ for all $x \in X$
- set $\alpha'(x, t) := \Sigma_{y \in X} \beta(x, y)$ for all $x \in X$
- set $\alpha'(x, y) := \alpha(x, y) - \beta(x, y)$ for all $(x, y) \in X \times X$

Example Network with Lower Bounds



Associated s-t-Network

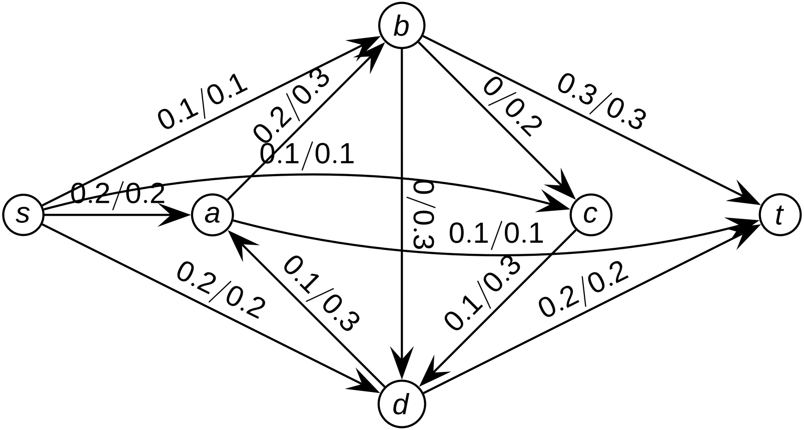


Existence Criterion for Circulations

Theorem

There is a circulation in a network with lower bounds $N = (\alpha : X \leftrightarrow X, \beta : X \leftrightarrow X)$ iff the value of a maximal flow in the associated s-t-network equals $|\beta|$.

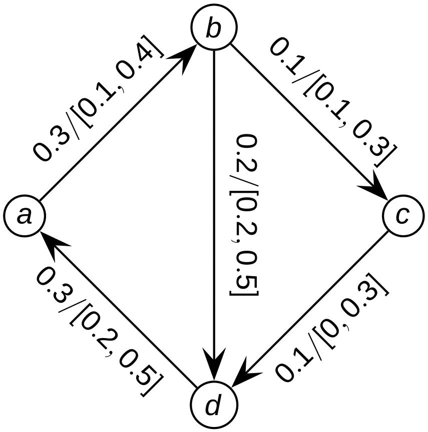
Maximal Flow in the Associated s-t-Network



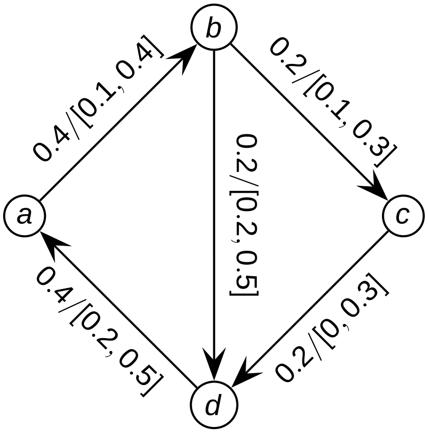
Reverting the construction of the associated s-t-network leads from a maximal flow φ' to a circulation φ:

- remove all edges out of s and leading into t
- set $\varphi(x,y) = \varphi'(x,y) + \beta(x,y)$ for all $(x,y) \in X \times X$

Resulting Circulation



Another Circulation



Summary

We saw

- an algebraic description of network problems
- application of semirings
- Kleene Algebra in working in background

To Do's

- applying similar methods to related problems
- describing graph theoretical problems with these methods
- partial automation of proofs

Thanks for Listening

Enjoy the Calanques!