Using Bisimulations for Optimality Problems in Model Refinement

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About

- dioid-based optimality problems
- model refinement
- bisimulations
- combination of all

- recent work by Michel Sintzoff (MPC 2008)
- algebraic characterisation of bisimulations at RelMiCS 2009
- generic algorithm at AMAST 2010

Basic Definition

Definition

A *complete dioid* is a structure $(D, \Sigma, 0, \cdot, 1)$ such that $(D, \sqsubseteq)$ is a complete lattice with supremum operator Σ and least element 0, where $\sqsubseteq$ is defined by $x \sqsubseteq y \Leftrightarrow \Sigma\{x, y\} = y$, $(D, \cdot, 1)$ is a monoid and $\cdot$ distributes over Σ from both sides. $\sqsubseteq$ is called the *order* of the complete dioid.

- naming dioid after Gondran/Minoux
- also known as quantale

Selective Dioids

- special case of *selective* dioids
- $a + b \in \{a, b\}$. i.e. $\sqsubseteq$ is linear
- abbreviation *s-dioid* for complete selective dioids
- e.g. $(\mathbb{R} \cup \{-\infty, \infty\}, \sup, -\infty, \inf, \infty)$, $(\mathbb{R} \cup \{-\infty, \infty\}, \inf, \infty, +, 0)$

Cumulative Dioids

- cumulative dioids
- characterised by $a \sqsubseteq 1$
- equivalent to:
  - $\forall a, b, c \in D : a \sqsubseteq b \Rightarrow ac \sqsubseteq b \wedge a \sqsubseteq b \Rightarrow ca \sqsubseteq b$
  - $\forall a, b \in D : ab \sqsubseteq a \wedge ba \sqsubseteq a$
- interpretation will be given soon

Definition of Models

- model: pair (G, g) where
  - $G = (V, E)$ is a graph
  - $g : E \to D$ is an edge labelling function
  - D is carrier set of an s-dioid
- target model: model with target set $T \subseteq V$ (and some additional technical requirements)

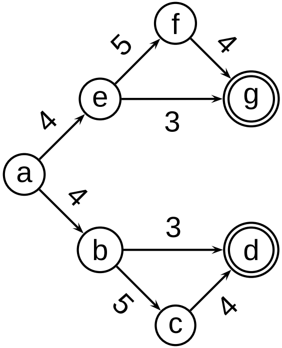
Costs in Models

- *cost* $c(w)$ of a walk $x_1x_2\dots x_n$ in $M=(G,g)$ defined by $c(w)=\prod_{i=1}^{n-1} g(x_i,x_{i+1})$
- *distance* $d(x,y)$ by $d(x,y)=\sum_{w\in W(x,y)} c(w)$
- *target distance* in a target model by $d(x)=\sum_{t\in T} d(x,t)$
- $x_1x_2\dots x_n$ is *optimal* walk if $c(x_1x_2\dots x_n)=d(x_1,x_n)$

Interpretation

- suitable choices of D yield different optimality problems
- $(\mathbb{R} \cup \{-\infty, \infty\}, \text{inf}, \infty, +, 0)$ corresponds to shortest path problem
- $(\mathbb{R} \cup \{-\infty, \infty\}, \text{sup}, -\infty, \text{inf}, \infty)$ corresponds to maximum capacity path
- application in routing, planning, optimisation, ...

Example



target set

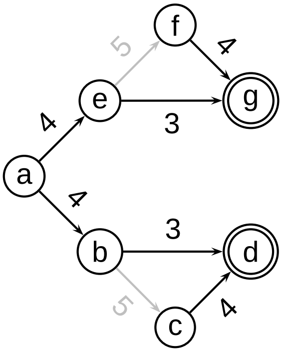
double surrounded

Optimal Submodels

- $((V,E'),g',T)$ is *target submodel* of $((V,E),g,T)$ if $E' \subseteq E$ and $g' = g|_{E'}$
- $((V,E'),g',T)$ is *optimal target submodel* of $((V,E),g,T)$ if
  - $((V,E'),g',T)$ is target submodel of $((V,E),g,T)$
  - all walks from arbitrary x into T are optimal
  - T is reachable from every node $x \in V - T$
- goal: *refine* given target model to an optimal target model

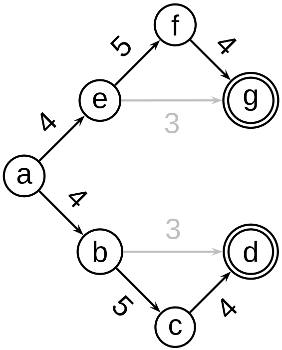
Refinment Algorithms

- refinment in case of cumulative dioids by Dijkstra-like algorithm
- key point: prolonging a path can not improve its cost
  - recall $ab \sqsubseteq a$
- in general case by Floyd-Warshall-like algorithm
- in the absence of negative cycles
  - cycles with cost $\sqsupseteq 1$



optimal submodel for

shortest paths



optimal submodel for

maximum capacity paths

Definition

$B \subseteq V_1 \times V_2$ is a *bisimulation* between two graphs (V_1, E_1) and (V_2, E_2) iff

- $Dom(B) = X_1$ and $Cod(B) = X_2$
- $v_1 B v_2 \wedge v_1 E_1 w_1 \Rightarrow \exists w_2 : w_1 B w_2 \wedge v_2 E_2 w_2$
- $v_2 B^{\smile} v_1 \wedge v_2 E_2 w_2 \Rightarrow \exists w_1 : w_2 B^{\smile} w_1 \wedge v_1 E_1 w_1$

relational definition:

- $B^{\smile}; E_1 \subseteq E_2; B^{\smile} \ \wedge \ R; E_2 \subseteq E_1; B$

additional requirement: respecting edge labels

Coarsest Bisimulation

- bisimulations between G and itself are closed under
  - union,
  - composition, and
  - taking the converse
- identity is a bisimulation between G and itself
- existence of a *coarsest bisimulation equivalence on* G

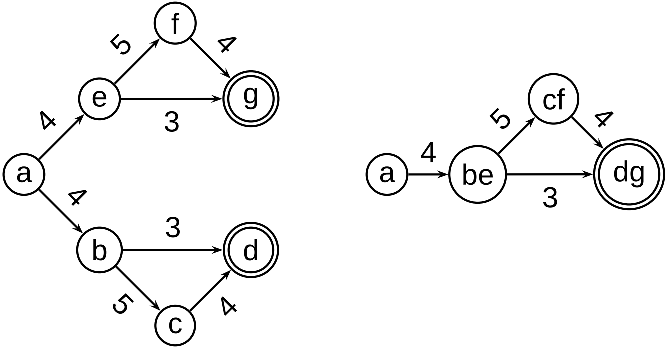
Compatible Bisimulations

- bisimulation equivalence B respects partition $V = \dot{\bigcup}_{i \in I} V_i$ if every V_i is the union of suitable equivalence classes of B
- for every partition of V exists a coarsest respecting bisimulation
- here main interest in bisimulations respecting $\{T, V - T\}$

Quotient Graph

- for a bisimulation equivalence B and a graph $G = (V, E)$ the *quotient* $G/B = (V/B, E/B)$ is defined by
  - V/B is the set of equivalence classes of B
  - $(v/B, w/B) \in E/B \Leftrightarrow (v, w) \in E$
- G/B has in general a smaller node set then G
- *coarsest quotient* respecting T induced by coarsest bisimulation respecting $\{T, V - T\}$

Example Quotient Graph



Algorithm Idea

- idea: refine the quotient and expand the solution
- missing: expansion operation
- solution: for a subgraph $(G/B)' = (V/B, (E/B)')$ of G/B define the *expansion* $(G/B)' \backslash B = (V', E')$ by
  - $V' = V$
  - $(v_1, v_2) \in E' \Leftrightarrow (v_1, v_2) \in E \wedge (v_1/B, v_2/B) \in (E/B)'$

Algorithm

given : target model M

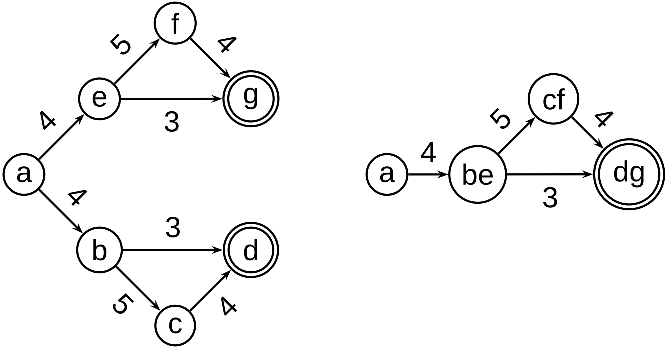
compute the coarsest quotient M/B of M

compute an optimal submodel $(M/B)'$ of M/B

compute the expansion $(M/B)' \backslash B$

output: $(M/B)' \backslash B$ is an optimal submodel of M

Example Execution for Shortest Paths



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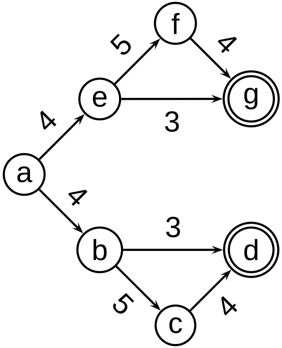

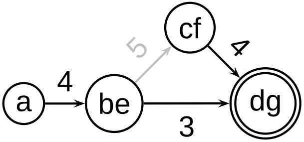

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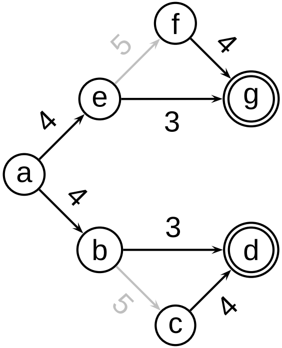

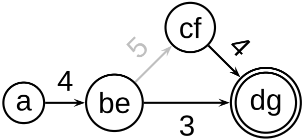

Runtime Considerations

- computation of coarsest bisimulation/quotient in $\mathcal{O}(|E| \cdot log(|V|))$ (Tarjan/Paige)
- runtime of Dijsktra-like refinement: $\mathcal{O}(|V| \cdot log(|V|) + |E|)$
- runtime of Floyd-Warshall-like refinement: $\mathcal{O}(|V|^3)$
- speed up possible for non-cumulative dioids
- depending on the structure of the model

Further Work

- hunting other problems solvable via bisimulations
- algebraic foundation
- application of ATPs