Algebraic Investigation of Connected Components

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Overview

- recent work
- our approach
- complementary closures
- graph algebras
- various kinds of connected components
- outlook

Recent Work

author(s)	theme	used structure
Tarski	basic principles	relation algebra
Schmidt/Ströhlein	overview	relation (linear) algebra
Kawahara/Furusawa	network flows/matchings	fuzzy relations
Berghammer/Höfner/Stucke	vertex coloring	relation algebra
Berghammer/Stucke/Winter	bipartions	relation algebra/ Dedekind categories
Kahl	graph transformation	category theory
this work	connected components	modal Kleene algebra ++

Modelling of Nodes

Author(s)	node modelling
Tarski	none
Schmidt/Ströhlein	implicit/vectors
Kawahara/Furusawa	point axiom
Berghammer/Höfner/Stucke	point vectors
Berghammer/Stucke/Winter	point vectors
Kahl	not explicit
this work	atomic tests

Full Atomic Lattices

- complete Boolean algebra $\mathcal{M} = (M, \sqsubseteq, \bigsqcup, \bigsqcap, \bot, \top, \bar{\ })$ defined as usual
- atoms are minimal non-bottom elements
- set of atoms denoted by atom($\mathcal{M}$)

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- atoms are minimal non-bottom elements
- set of atoms denoted by $\mathrm{atom}(\mathcal{M})$

Definition (Full Atomic Lattice)

A complete Boolean algebra $\mathcal{M} = (M, \sqsubseteq, \bigsqcup, \bigsqcap, \bot, \top, \bar{\ })$ is called a *full atomic lattice* if $m = \bigsqcup \{m^a \in \mathrm{atom}(\mathcal{M}) \mid m^a \sqsubseteq m\}$ holds for all $m \in M$.

- will serve as model for carrier set of relation algebra

Complementary Distributive Closures

Definition (Complementary Distributive Closure)

Let $\mathcal{M} = (M, \sqsubseteq, \bigsqcup, \bigsqcap, \bot, \top, \bar{\ })$ be a full atomic lattice. A function $f : M \to M$ is called a *complementary distributive closure* (cd closure) if the following holds:

- $m \sqsubseteq f(m)$ for all $m \in M$ *(extensivity)*
- $f(f(m)) = f(m)$ for all $m \in M$ *(idempotence)*
- $f(\bigsqcup M') = \bigsqcup f(M')$ for all $M' \subseteq M$ *(distributivity)*
- $f(\overline{f(m)}) = \overline{f(m)}$ for all $m \in M$ *(complementary idempotence)*

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- f is isotone, so fix_f is a complete lattice (Knaster-Tarski)
- fix_f is closed under complementation
- $\bot \in \text{fix}_f$, $\top \in \text{fix}_f$
- $\mathcal{FIX}_f =_{df} (\text{fix}_f, \sqsubseteq, \bigsqcup, \bigsqcap, \bot, \top, \bar{\ })$ is complete Boolean algebra
- $\mathcal{FIX}_f$ is even a full atomic lattice with $\text{atom}(\mathcal{FIX}_f) = f(\text{atom}(\mathcal{M}))$

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Quantales

Definition (Quantale)

$\mathcal{Q} = (M, \sqsubseteq, \bigsqcup, \bigsqcap, \bot, \top, 1, \cdot)$ is a *quantale* if

- $(M, \sqsubseteq, \bigsqcup, \bigsqcap, \bot, \top)$ is a complete distributive lattice,
- $\cdot : M \times M \to M$ (the *multiplication*) is associative with neutral element 1,
- $\cdot$ distributes over arbitrary suprema and, additionally,
- $m \neq \bot \Leftrightarrow \top \cdot m \cdot \top = \top$ holds for all $m \in M$. *(Tarski rule)*

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- endorelations over a fixed set form a quantale with union as $\bigsqcup$ and composition as multiplication
- elements p with a relative complement $\neg p$ (i.e. $p \sqcup \neg p = 1$ and $p \cdot \neg p = \bot = \neg p \cdot p$) are called *tests*
- correspond to subidentities/subsets of the carrier set

Graph Algebras (Definition)

Definition (Graph Algebra)

$\mathcal{GA} = (M, \sqsubseteq, \bigsqcup, \bigsqcap, \bot, \top, \cdot, 1, ^*, ^\circ, |\rangle, \langle|)$ is called a *graph algebra* if

- $\mathcal{Q} =_{df} (M, \sqsubseteq, \bigsqcup, \bigsqcap, \bot, \top, \cdot, 1)$ is a quantale
- $\mathbf{test}(\mathcal{Q})$ is a full atomic lattice
- $|\rangle, \langle| : M \times \mathbf{test}(\mathcal{Q}) \to M$ are defined by $|m\rangle p \sqsubseteq q \Leftrightarrow \neg q m p \sqsubseteq \bot \Leftrightarrow \langle m|\neg q \sqsubseteq \neg p$
- $|mn\rangle p = |m\rangle|n\rangle p$ and $\langle mn|p = \langle n|\langle m|p$ *(modality)*
- $^\circ : M \to M$ obeys $|m^\circ\rangle p = \langle m|p$ for all $m \in M$ and tests p *(converse)*
- $p^a m q^a = \bot \Leftrightarrow p^a m q^a \neq p^a \top q^a$ for atomic tests p^a, q^a *(all-or-nothing)*
- $^* : M \to M$ is characterized by
  - $1 \sqcup mm^* \sqsubseteq m^*$ and $1 \sqcup m^* m \sqsubseteq m$ *(star unfold)*
  - $n \sqcup mo \sqsubseteq o \Rightarrow m^* n \sqsubseteq o$ and $n \sqcup om \sqsubseteq o \Rightarrow nm^* \sqsubseteq o$ *(star induction)*

Graph Algebras (Interpretation)

- *forward* and *backward diamond* $|\rangle$ and $\langle|$ correspond to preimage and image
- $^\circ$ models relational converse
- all-or-nothing means unlabelled graphs
- *Kleene star* * corresponds to iteration/reflexive-transitive hull
- sets of vertices correspond to tests
- single vertices can be described by atomic tests ($\mathbf{att}(\mathcal{GA})$)
- a lot of natural implications as
  - converse distributes over arbitrary suprema and infima
  - $(m^*)^\circ = (m^\circ)^*$, $(m^\circ)^\circ = m$
  - $m = n \Leftrightarrow \forall p \in \mathbf{test}(\mathcal{GA}) : \langle m|p = \langle n|p \Leftrightarrow \forall p \in \mathbf{test}(\mathcal{GA}) : |m\rangle p = |n\rangle p$
  - $p^a mnq^a \neq \bot \Leftrightarrow \exists r^a \in \mathbf{att}(\mathcal{GA}) : (p^a mr^a \neq \bot \wedge r^a nq^a \neq \bot)$
- undirected graphs can be modelled by $|m\rangle = \langle m|$, i.e., m is symmetric

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Algebraic Equivalences

Definition (Algebraic Equivalence)

An *algebraic equivalence* is an element g of a graph algebra with

- $1 \sqsubseteq g$ *(reflexivity)*
- $gg \sqsubseteq g$ *(transitivity)*
- $g^\circ = g$ *(symmetry)*

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Theorem

Let g be an algebraic equivalence. Then $\langle g|$ is a complementary distributive closure on $\mathbf{test}(\mathcal{GA})$.

Innately Connected Components

Definition (Innately Connected Component)

Given an algebraic equivalence g, a test p is said to be *innately g-connected* if $p_2 \sqsubseteq \langle g | p_1$ holds for all $\bot \sqsubset p_1, p_2 \sqsubseteq p$. An *innately connected component* of g is an innately g-connected test p such that every test q with $q \sqsupset p$ is not innately g-connected.

- set of innately connected components of g denoted by $\mathrm{icc}(g)$
- $\bot \in \mathrm{fix}_{\langle g|}$ but $\bot \notin \mathrm{icc}(g)$ in the nontrivial case

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- set of innately connected components of g denoted by $\mathrm{icc}(g)$
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Theorem

Let g be an algebraic equivalence. Then $(\mathrm{fix}_{\langle g|}, \sqsubseteq, \bigsqcup, \bigsqcap, \bot, 1, \neg)$ *is a full atomic lattice. Its atoms are exactly* $\mathrm{icc}(g)$*.*

- "proof": $\langle g|$ is an cd closure on $\mathbf{test}(\mathcal{GA})$. ■

Connected Components in Undirected Graphs

Definition (Connected Component)

Given a symmetric g, a test p is said to be *g-connected* if $p_2 \sqsubseteq \langle g^* | p_1$ holds for all $\bot \sqsubset p_1, p_2 \sqsubseteq p$. A *connected component* of g is a g-connected test p such that every test q with $q \sqsupset p$ is not g-connected.

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- hence: connected components of g have the form $\langle g^* | p^a$ with atomic test p^a
- algorithmically by BFS/DFS

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Theorem

Let g be symmetric and consider two connected components c_1 and c_2 of g with $c_1 \neq c_2$. Then $c_1 g c_2 = \bot$ holds.

Connected Components in Directed Graphs (Definition and Basics)

Definition (Strongly Connected Component)

Given an arbitrary g, a test p is said to be *strongly g-connected* if $p_2 \sqsubseteq \langle g^* | p_1$ holds for all $\bot \sqsubset p_1, p_2 \sqsubseteq p$. A *strongly connected component* of g is a strongly g-connected test p such that every test q with $q \sqsupset p$ is not strongly g-connected.

- Caveat: in general, $\langle g^* |$ is no distributive closure!
- Notation: scc(g) for set of stronlgy connected components of g

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Lemma

Let p be strongly g-connected. Then $pg^*p = p\top p = p(g^\circ)^*p$ holds.

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Lemma

Let p be strongly g-connected. Then $pg^*p = p\top p = p(g^\circ)^*p$ holds.

Lemma

p is strongly g-connected iff for all non-bottom tests $p_1, p_2 \sqsubseteq p$ the inequality $p_2 \sqsubseteq \langle g^* \sqcap (g^\circ)^* | p_1$ holds.

Connected Components in Directed Graphs (Fixpoint Charaterisation)

Theorem

For all g, the element $g^* \sqcap (g^\circ)^*$ *is an algebraic equivalence.*

Connected Components in Directed Graphs (Fixpoint Charaterisation)

Theorem

For all g, the element $g^* \sqcap (g^\circ)^*$ is an algebraic equivalence.

- now theorem about innately connected components applicable
- strongly connected components of g are atomic fixpoints of $\langle g^* \sqcap (g^\circ)^* |$
- algorithmically unsatisfactory

Algorithm Sketch for Strongly Connected Components

Algorithm given by Sharir (1981):

- graph induced by strongly connected components of $G = (V, E)$ is acyclic
- chose a node v_s in a sink of this graph (possible if $|V| < \infty$)
- $C_1 = v_s E^*$ is a strongly connected component of G
- remove C_1 and proceed recursively
- computation of v_s by means of DFS

Acyclic Graphs and Sinks

Definition (Algebraic Acyclic Directed Graph)

An element $g \in \mathcal{GA}$ is called an *algebraic directed acyclic graph (algebraic dag)* if $g^+ \sqcap 1 = \bot$ holds (with $g^+ \;=_{df}\; gg^* = g^*g$).

Definition (Algebraic Sink)

Given an arbitrary $g \in \mathcal{GA}$, a non-bottom test s is called an *algebraic sink* of g if $\langle g | s = \bot$ holds.

Theorem

Every algebraic dag in a finite graph algebra has an algebraic sink.

- proof uses explicit formula $g^* = \bigsqcup_{n=0}^{\infty} g^n$
- seems unavoidable since theorem is false for infinite graph algebras

Component Graph

Theorem

For an arbitrary g, the component graph $scg_g =_{df} \bigsqcup\{c_1 g c_2 \mid c_1, c_2 \in scc(g), c_1 \neq c_2\}$ *is an algebraic dag.*

Theorem

Let g be an arbitrary element of a finite graph algebra and consider a sink c_s *of* scg_g*. Then for every atomic test* $p^a \sqsubseteq c_s$ *we have* $\langle g^* | p^a = c_s$*.*

- shows two pillars of Sharir's algorithm
- completion remains future work

Conclusion and Outlook

- graph algebras offer a suitable framework
- desription and analysis of connected components satisfactory
- desription and analysis of strongly connected components seems on a good way
- algebraic investigation of DFS
- final goal: automated/interactive verification