A Matrix-oriented View of Bisimulation Quotients over Dioid-labeled Transition Systems

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Used Structures

- matrices - known
- labeled transition systems - known
- dioid: $(\Delta, \oplus, 0, \otimes, 1)$ with order $\sqsubseteq$ $(x \sqsubseteq y \Leftrightarrow x \oplus y = y)$
- here: transition systems over complete graphs with unique edge labels drawn from a dioid Δ
- modeled by matrices over Δ
- $\oplus$, $\otimes$ and $\sqsubseteq$ extended to matrices over Δ in the usual way
- 0-1-matrices: all entries either 0 or 1
- can be seen as relations over the index/node set
- usual concepts like injectivity, transitivity, ... also used for 0-1-matrices

Why?

- least fixpoint of $Ax \oplus b = x$ (or transposed version) models
  - shortest paths ($\Delta = (\mathbb{R} \cup \{+\infty\}, \min, +, +\infty, 0)$)
  - maximum capacity paths ($\Delta = (\mathbb{R} \cup \{\pm\infty\}, \max, \min, -\infty, +\infty)$)
  - Bellman-Ford equations
  - regular languages (automaton, language semiring)
  - b corresponds to start/terminal states
- eigenvectors/eigenvalues used for
  - hierarchical clustering
  - preference analysis
- see Gondran/Minoux for an extensive overview

Definition of Equivalence Decomposition

Definition

A 0-1-matrix $E \in \Delta^{n \times n}$ is called an *equivalence* if E is reflexive ($\mathrm{I}^n \sqsubseteq E$), transitive ($EE \sqsubseteq E$), and symmetric ($E^t = E$).

Theorem

Let $E \in \Delta^{n \times n}$ be an equivalence. Then there is a (unique) $m \leq n$ and a surjective (left-total) function $D \in \Delta^{n \times m}$, called an equivalence decomposition of E, such that $DD^t = E$.

Intuition behind Equivalence Decomposition

- equivalences induce a partition of $\{1 \dots n\}$ (i.e., the node set of a transition system)
- equivalence classes can be labeled by numbers in $\{1 \dots m\}$
- $D_{ij} = 1 \Leftrightarrow$ node i lies in equivalence class j
- not unique:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Traditional Bisimulations Revisited

- consider (classical) transition systems $\to_1 \subseteq V \times \Sigma \times V$ and $\to_2 \subseteq W \times \Sigma \times W$
- a left- and right-total relation $B \subseteq V \times W$ is a bisimulation between $\to_1$ and $\to_2$ if it fulfills the following properties:
  - $\forall \sigma \in \Sigma : v_1 \to_1^\sigma v_2 \land v_1 B w_1 \Rightarrow \exists w_2 : w_1 \to_2^\sigma w_2 \land v_2 B w_2$
  - $\forall \sigma \in \Sigma : w_1 \to_2^\sigma w_2 \land w_1 B^\cup v_1 \Rightarrow \exists v_2 : v_1 \to_1^\sigma v_2 \land w_2 B^\cup v_2$
- algebraic formulation:
  - $\forall \sigma \in \Sigma : B^\cup ; \to_1^\sigma \subseteq \to_2^\sigma ; B^\cup$
  - $\forall \sigma \in \Sigma : B ; \to_2^\sigma \subseteq \to_1^\sigma ; B$

Bisimulations

Definition

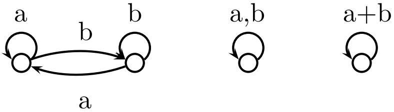
A 0-1-matrix $S \in \Delta^{n \times m}$ is called a *bisimulation* between two matrices $A \in \Delta^{m \times m}$ and $B \in \Delta^{n \times n}$, written $A \sim_S B$, if the two inequalities $SA \sqsubseteq BS$ and $S^t B \sqsubseteq AS^t$ hold.

- $A \sim_{S_i} B \Rightarrow A \sim_{(\bigoplus_{i \in I} S_i)} B$
- $A \sim_{S_1} B \wedge B \sim_{S_2} C \Rightarrow A \sim_{(S_1 \otimes S_2)} C$
- $A \sim_S B \Leftrightarrow B \sim_{S^t} A$
- $A \sim_{\mathrm{I}^n} A$ for $A \in \Delta^{n \times n}$

Definition

- backward bisimulation for A is bisimulation for A^t
- full bisimulation for A is bisimulation + backward bisimulation for A

Multiple Labels vs. Unique Labels



$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} a+b \end{pmatrix} = \begin{pmatrix} a+b \\ a+b \end{pmatrix} = \begin{pmatrix} a & b \\ a & b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} a & b \\ a & b \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix} \sqsubseteq \begin{pmatrix} a+b & a+b \end{pmatrix} =$$
$$\begin{pmatrix} a+b \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Bisimulation Equivalences

Definition

An equivalence E which is a bisimulation between A and itself is called a *bisimulation equivalence*.

- amounts to $EA \sqsubseteq AE$
- closed under sum
- existence of a greatest (wrt. $\sqsubseteq$) bisimulation equivalence for A
- known concept from automata theory (equivalence, minimality)

Bisimulation Quotient

Definition

Let $E \in \Delta^{n\times n}$ be a bisimulation equivalence for $A \in \Delta^{n\times n}$, and let $D \in \Delta^{n\times m}$ be an equivalence decomposition of E. Then the *quotient of A by D* is defined by $A/D =_{def} D^t AD$.

Intuition:

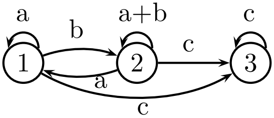
- node set of A/D corresponds to equivalence classes of E
- edge between two nodes of A/D is labeled by sum of all edge labels in A between nodes of the respective equivalence classes
- (in the classical setting set of all such labels)

Theorem

A and A/D are bisimilar.

caveat: not every bisimulation equivalence corresponds to a classical one!

Bisimulation Equivalence with Decomposition



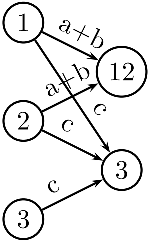
$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} a & b & c \\ a & a+b & c \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} a & a+b & c \\ a & a+b & c \\ 0 & 0 & c \end{pmatrix} \sqsubseteq$$

$$\begin{pmatrix} a+b & a+b & c \\ a+b & a+b & c \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} a & b & c \\ a & a+b & c \\ 0 & 0 & c \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

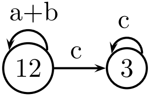
$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Quotient System

$$\begin{pmatrix} a & b & c \\ a & a+b & c \\ 0 & 0 & c \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+b & c \\ a+b & c \\ 0 & c \end{pmatrix}$$



$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} a+b & c \\ a+b & c \\ 0 & c \end{pmatrix} = \begin{pmatrix} a+b & c \\ 0 & c \end{pmatrix}$$



Compatibility

DLR

Definition

Let $E \in \Delta^{n \times n}$ be an equivalence, and let b be a column vector with $Eb = b$. Then b is called *compatible with* E.

- analogous definition for row vectors
- b is compatible with $E \Leftrightarrow (E_{ij} = 1 \Rightarrow b_i = b_j)$

Quotient and Expansion of Vectors

Definition

Let $D \in \Delta^{n \times m}$ be an equivalence decomposition of $E \in \Delta^{n \times n}$, and let b be a column (row) vector compatible with E. Then the *quotient of b by D* is defined by $b/D =_{def} D^t b$ ($b/D =_{def} bD$).

- indices of b correspond to equivalence classes of E
- entries of b in same equivalence classes are compressed in b/D

Definition

Let $D \in \Delta^{n \times m}$ be an equivalence decomposition of $E \in \Delta^{n \times n}$, and let $\hat{b}$ be a column (row) vector with m entries. Then the *expansion of $\hat{b}$ by D* is defined by $\hat{b} \backslash D =_{def} D\hat{b}$ ($\hat{b} \backslash D =_{def} \hat{b} D^t$).

- $(\hat{b} \backslash D)/D = \hat{b}$
- $(b/D) \backslash D = b$ (if compatible)

Solving Problems via Quotients

- in general, A/D is smaller then A
- idea:
  - solve problem for quotients
  - propagate solution back to original problem
- derive solution of $Ax = b$ from solution of $(A/D)\hat{x} = b/D$
- derive solution of $Ax + b = x$ from solution of $(A/D)\hat{x} + b/D = \hat{x}$
- derive solution of $Ax = \lambda x$ from solution of $(A/D)\hat{x} = \hat{x}$

Linear Equations

Theorem

Let E be a full bisimulation equivalence for A, let b be a column vector compatible with E, and let D be an equivalence decomposition of E. Then there exists a column vector x satisfying $Ax = b$ iff there exists a column vector $\hat{x}$ satisfying $(A/D)\hat{x} = b/D$. Moreover, under these conditions, we have $A(\hat{x}\backslash D) = b$.

- all conditions (fullness, compatibility necessary)
- counterexamples by Mace4

proof sketch: $Ax = b$ implies existence of x' compatible with E satisfying $Ax' = b$, rest by simple application of definitions and calculation

Linear Fixpoints

Theorem

Assume that Δ is a complete dioid. Let E be a bisimulation equivalence for A, let D be qn equivalence decomposition of E, and let b be a column vector compatible with E. Denote by x_μ the least solution of the equation $Ax \oplus b = x$, and denote by $\hat{x}_\mu$ the least solution of $(A/D)\hat{x} \oplus b/D = \hat{x}$. Then the equality $x_\mu = \hat{x}_\mu \backslash D$ holds.

proof sketch: iterate $f(x) = Ax \oplus b$ and $\hat{f}(\hat{x}) = (A/D)\hat{x} \oplus b/D$ starting at zero vector for x and $\hat{x}$. Simple induction shows $f^i(x) = \hat{f}^i(\hat{x})\backslash D$, apply Knaster-Tarksi.

Eigenvectors and Eigenvalues

Definition

x is called *right (left) eigenvector of* A for the eigenvalue λ if $x \neq 0$ and $Ax = \lambda x$ ($xA = x\lambda$) hold.

Lemma

Let A, E, D *be as always, and let* x *be a right eigenvector of* A *for the eigenvalue* λ*, and let* x *be compatible with* E*. Then* x/D *is a right eigenvector of* A/D *for the eigenvalue* λ*.*

Lemma

Let A, E, D *be as always, and let* $\hat{x}$ *be a right eigenvector of* A/D *for the eigenvalue* λ*. Then* $\hat{x} \backslash D$ *is a right eigenvector of* A *for the eigenvalue* λ*.*

- analogous claims for left eigenvectors do not hold!

Open Problems

- wanted: greatest bisimulation equivalence for A (+ compatibility)
- every classical bisimulation equivalence induces bisimulation equivalence (+ compatibility)
- partition refinement algorithm by Tarjan/Paige
- can we do better?
- presented approach avoids "point chasing" style of proofs
- is it possible to handle multiple edge labels?
- how to extend approach to infinite systems?

Thanks for Listening

Questions?