A Semiring Approach to Equivalences, Bisimulations and Control

Roland Glück[1] Bernhard Möller[1] Michel Sintzoff[2]

[1]Universität Augsburg

[2]Université catholique de Louvain

Nov 3, 2009
Doha

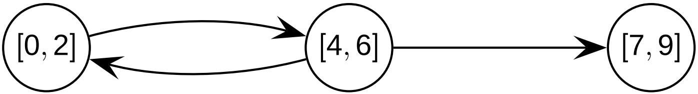
Outline

- introducing example
- algebraic theory
- application
- further work

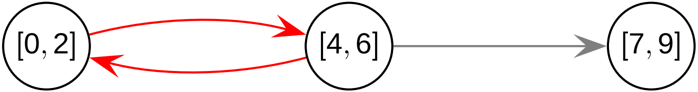
Example Problem

- given: relation $T \in \mathbb{R} \times \mathbb{R}$ with $xTy \Leftrightarrow$
  - $x \in [0,2] \wedge y = x + 4 \vee$
  - $x \in [4,6] \wedge y = x + 3 \vee$
  - $x \in [4,6] \wedge y = x - 4$
- goal: determine greatest life part of T, i.e.
- greatest relation $T' \subseteq T$ with $T'x \neq \emptyset \Rightarrow xT' \neq \emptyset$

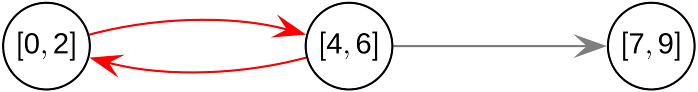
Solution



Solution



Solution



Solution: $\{(x, x+4) \mid x \in [0,2]\} \cup \{(x, x-4) \mid x \in [4,6]\}$

Solution

Idea:

- merge numbers into suitable equivalence classes
- equivalence classes of a greatest bisimulation
- this produces a finite graph
- apply any algorithm to find the greatest life part
- transform this solution back to original problem

Goals

- embedding bisimulations into the framework of semirings/quantals/Kleene algebras
- similar approach to equivalences and partitions
- still work in progress!

Basics

Definition

A *quantal* S is a structure $(M, \leq, \cdot, 1, 0)$ with the following properties:

- $\leq$ is a partial order on M
- for arbitrary subsets $M' \subseteq M$ the supremum exists
- denoted by $\sum_{m \in M'} m$ or $m_1 + m_2$ (addition)
- 0 is the smallest element
- $(M, \cdot, 1)$ is a multiplicative monoid
- $0 \cdot m = 0 = m \cdot 0$ for all $m \in M$
- $\cdot$ distributes over arbitrary suprema

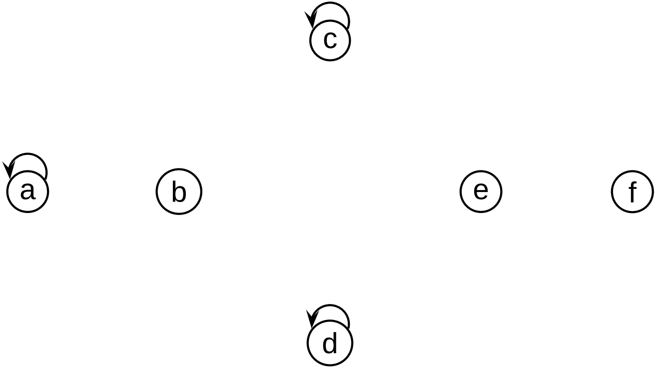
Example Quantals

- relations over a set M form a quantale with
  - $\subseteq$ as order
  - $\cup$ as addition
  - composition as multiplication
- subsets modelled via tests
- in relation algebra subrelations of the identity
- atomic tests model singleton elements

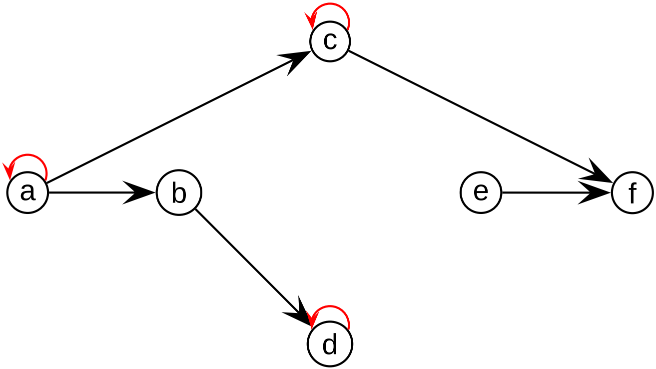
Using Tests

- tests form a boolean subalgebra
- multiplication of test corresponds to intersection
- negation ($\neg$) corresponds to complementation
- multiplication with tests models restrictions
- usually denoted by p, q, p_1, q', ...

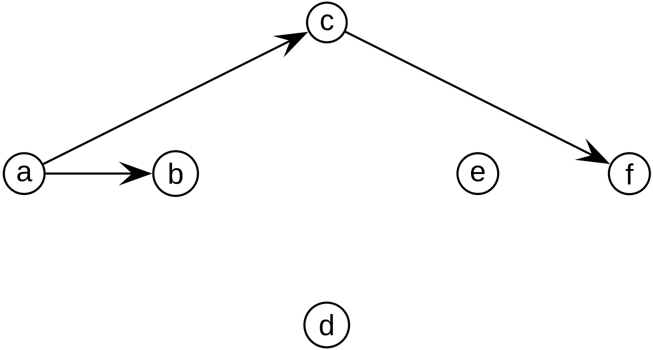
Example Usage



Example Usage



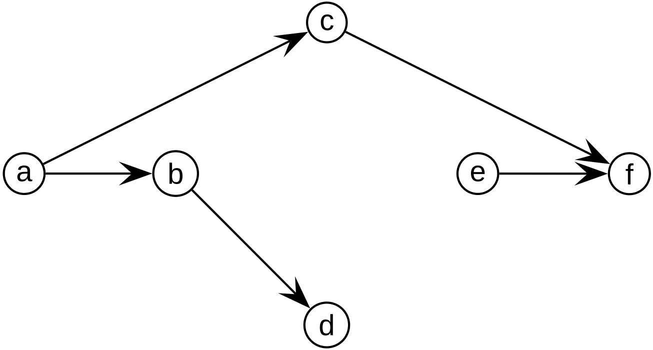
Example Usage



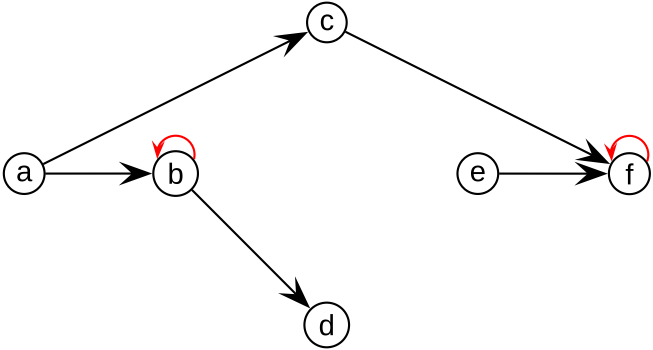
Diamond Operators

- diamond operators are used for modelling domain/codomain
- forward diamond models preimage, backward diamond image
- notation:
  - $x \in M, p \in \mathrm{TEST}(S)$
  - $|x\rangle p$ for forward diamond
  - $\langle x|p$ for backward diamond
- $|x\rangle q \leq \neg p \Leftrightarrow p \cdot x \cdot q \leq 0 \Leftrightarrow \langle x|p \leq \neg q$

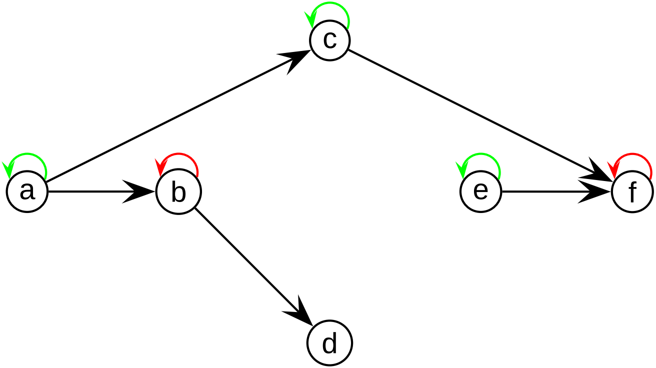
Example Forward Diamond



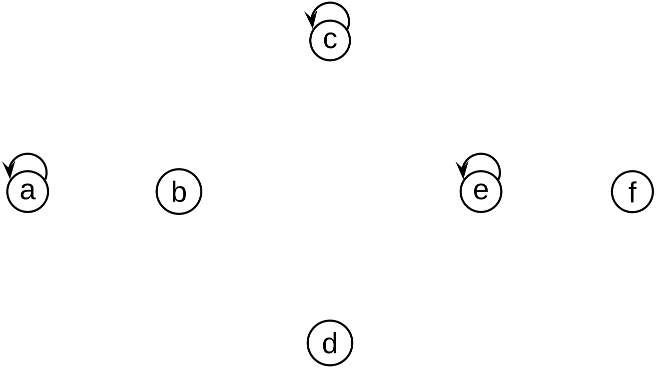
Example Forward Diamond



Example Forward Diamond



Example Forward Diamond



Definition Partition

Definition

A subset $\mathcal{P} \subseteq \mathrm{TEST}(S)$ is called a *partition*, if the following two requirements hold:

- $\sum_{p\in\mathcal{P}} p = 1$
- $p \neq q \Leftrightarrow p \cdot q = 0$

- natural generalisation of set theoretic definition
- $\{1\}$ is a partition
- atomic tests form a partition

Towards Equivalences

- an equivalence relation is a reflexive, transitive and symmetric relation
- modelling of reflexivity, transitivity and symmetry needed

Towards Equivalences

- an equivalence relation is a reflexive, transitive and symmetric relation
- modelling of reflexivity, transitivity and symmetry needed

Definition

- $x \in M$ is called *reflexive*, iff $1 \leq x$
- $x \in M$ is called *transitive*, iff $x \cdot x \leq x$
- $x \in M$ is called *symmetric*, iff $|x\rangle = \langle x|$

Definition Equivalence

Definition

An element $e \in M$ is called an *equivalence*, iff it is reflexive, transitive and symmetric.

- 1 is the smallest equivalence in S
- $\top$ is the greatest equivalence in S

Definition Equivalence Classes

- traditional definition of equivalence classes uses elements of the carrier set
- other approach: via fixed points

Definition Equivalence Classes

- traditional definition of equivalence classes uses elements of the carrier set
- other approach: via fixed points

Definition

For an equivalence $e \in M$ the set of *equivalence classes of* e is the set of atomic fixed points of $f(x) = \|e\rangle x$. It is denoted by $\mathcal{P}_e$.

Properties Equivalence Classes

- for all equivalences $e \in M$ the equivalence classes of e form a partition (induced partition)
- equivalence classes of 1 are the set of atomic tests
- equivalence class of $\top$ is $\{1\}$

Properties Equivalence Classes

- for all equivalences $e \in M$ the equivalence classes of e form a partition (induced partition)
- equivalence classes of 1 are the set of atomic tests
- equivalence class of $\top$ is $\{1\}$

Theorem

Let $p \in \mathrm{TEST}(M)$ be an atomic test and $e \in M$ be an equivalence. Then $|e\rangle p$ is a fixed point of $f(x) = |e\rangle x$, i.e. it is an equivalence class of e.

Induced Equivalences

- equivalences induce partitions
- partition $\mathcal{P}$ induces equivalence $\sum_{p \in \mathcal{P}} p \cdot \top \cdot p$
- induced equivalences and induced partitions form a Galois Connection

Previous Definitions

traditional definition:
$R \subseteq X_1 \times X_2$ is bisimulation between two transition systems $(X_1, \rightarrow_1)$ and $(X_2, \rightarrow_2)$ iff

- $Dom(R) = X_1$ and $Cod(R) = X_2$
- $x_1 R x_2 \wedge x_1 \rightarrow_1 y_1 \Rightarrow \exists y_2 : y_1 R y_2 \wedge x_2 \rightarrow_2 y_2$
- $x_2 R^{\smile} x_1 \wedge x_2 \rightarrow_2 y_2 \Rightarrow \exists y_1 : y_2 R^{\smile} y_1 \wedge x_1 \rightarrow_1 y_1$

relational definition:

- $R^{\smile} ; \rightarrow_1 \subseteq \rightarrow_2 ; R^{\smile} \ \wedge \ R ; \rightarrow_2 \subseteq \rightarrow_1 ; R$

Bisimulations in Quantales

- here main interest in autobisimulations
- idea: $|x\rangle = \langle x^{\smile}|$ and $|x^{\smile}\rangle = \langle x|$

Bisimulations in Quantales

- here main interest in autobisimulations
- idea: $|x\rangle = \langle x^{\smile}|$ and $|x^{\smile}\rangle = \langle x|$

Definition

For an element $g \in M$ an element $b \in M$ is called a *bisimulation on g* iff $|b\rangle|g\rangle \leq |g\rangle|b\rangle$ and $\langle b||g\rangle \leq |g\rangle\langle b|$ hold. The set of all bisimulations on g is denoted by bisim_g.

Properties of Bisimulations

- 1 is a bisimulation on every $g \in M$
- bisimulations on an element $g \in M$ are closed under addition and multiplication
- so $\sum_{b \in \mathrm{bisim}_g} b =: \mathrm{gbs}_g$ is the greatest bisimulation on g

Properties of Bisimulations

- 1 is a bisimulation on every $g \in M$
- bisimulations on an element $g \in M$ are closed under addition and multiplication
- so $\sum_{b \in \mathrm{bisim}_g} b =: \mathrm{gbs}_g$ is the greatest bisimulation on g

Theorem

For an element $g \in M$ the greatest bisimulation on g is an equivalence.

- remark: holds only under slightly stronger conditions

- g is *live* if $\forall p \in \mathrm{TEST} |g\rangle p \neq 0 \Rightarrow \langle g| p \neq 0$
- 0 is live
- live elements are closed under union
- so greatest live part $g' \leq g$ exists
- algebraically determinable via bisimulation-based construction

Reducing the state number

- given a huge transition system
- goal: generating a policy (decision rule) for desired control objective (e.g. minimality of costs, termination, ...)
- idea: reduce the state number by generating equivalence classes of a suitable bisimulation
- solve the problem on this instance
- transform the solution back to the original instance

Plans

- two directions of future work:
- specialisation: deriving/verifying specific algorithms
- generalisation: developing a framework for a large class of problems (bisimulation formulae)
- automated reasoning, advantage of one-sorted algebra