Covering Polygons with Rectangles

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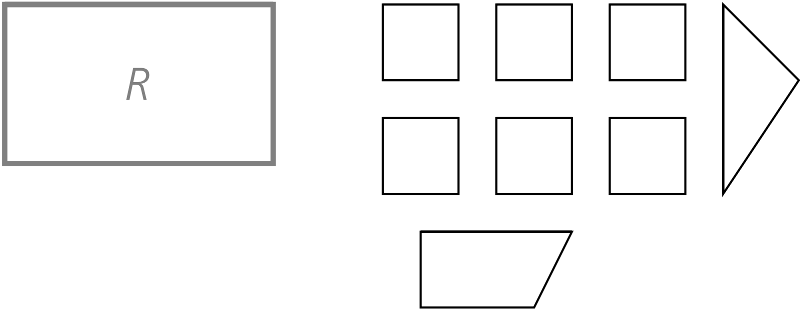
Motivation: Cutting Center

Basic Definitions

Definition

- *Nesting* $\mathbf{P} = \{P_1, P_2, \ldots, P_n\}$ is a set of n simple polygons.
- *Cover* $\mathbf{C} = \{R_1, R_2, \ldots, R_m\}$ of a nesting $\mathbf{P}$ by a gripper R is a set of rectangles such that
  - every R_j is a copy of R, and
  - every polygon $P_i \in \mathbf{P}$ is contained completely in at least one $R_j \in \mathbf{C}$.
- Problem: find an *optimal cover*, i.e. a cover with minimum cardinality
- Copies of R arise from R by
  - translation along the x-axis (easy to solve)
  - arbitrary translation (NP-hard)
  - arbitrary translation and rotation (NP-hard, considered here)

Example Cover



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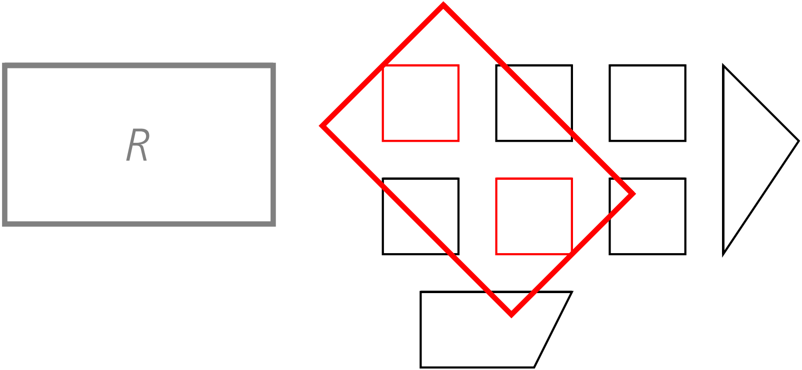

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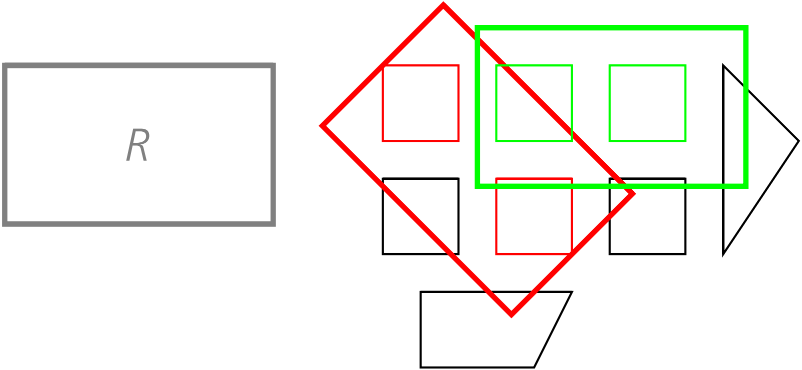

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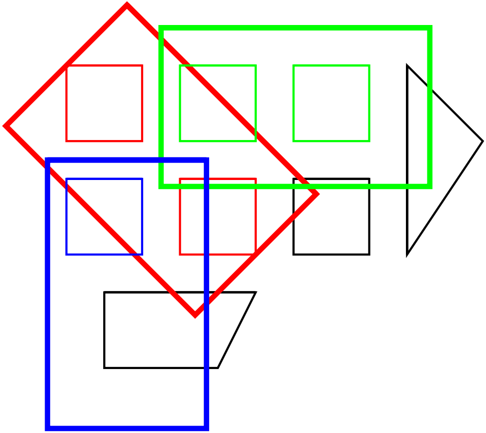

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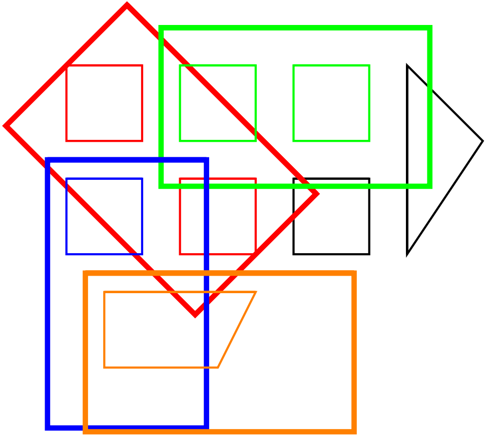

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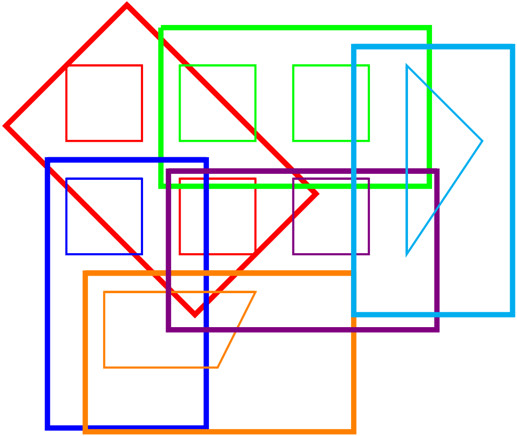
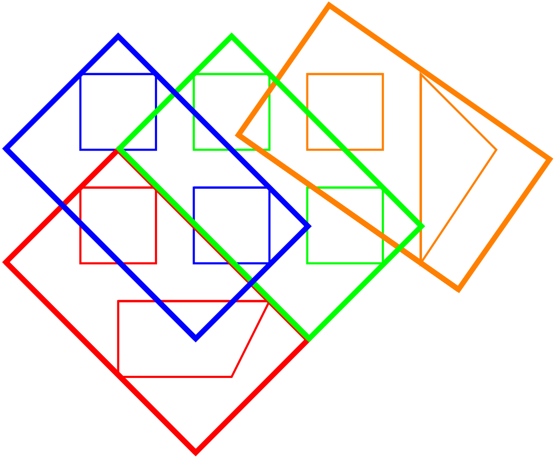
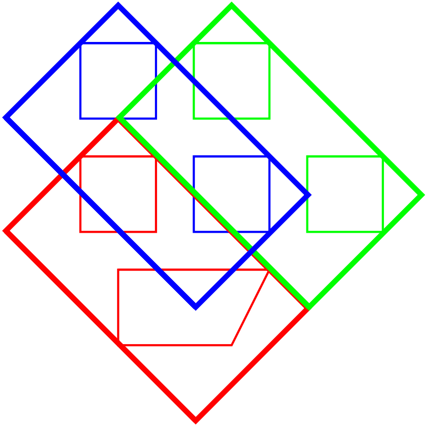


Optimal Cover



Decomposition of an Optimal Cover



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- Notation: given a nesting **P** and a rectangle R, we denote by $\mathrm{cov}(R, \mathbf{P})$ the set of all polygons of **P** which are contained completely in R.

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Lemma

Let $\mathbf{C} = \{R_1, R_2, \ldots, R_m\}$ be an optimal cover of $\mathbf{P} = \{P_1, P_2, \ldots, P_n\}$ and choose an arbitrary $R_j \in \mathbf{C}$. Then $\mathbf{C} \backslash \{R\}$ is an optimal cover of $\mathbf{P} \backslash \mathrm{cov}(R, \mathbf{P})$.

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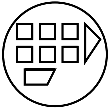
Work with a *set* of **candidate rectangles** containing at least one rectangle from an optimal cover.

- **Idea:** Every polygon has to be contained completely in at least one rectangle.
- Choose pivot polygon P_{pi} and compute $\subseteq$-maximal subsets of $\mathbf{P}$ containing P_{pi} which are contained in a copy of R.

Search Graph



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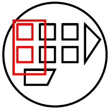

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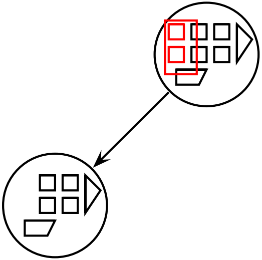

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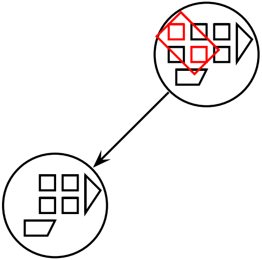
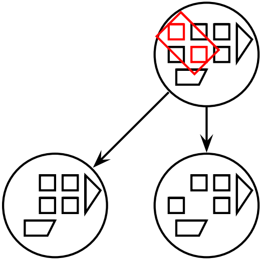


DLR

Search Graph



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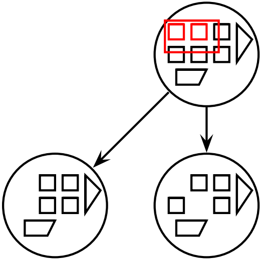

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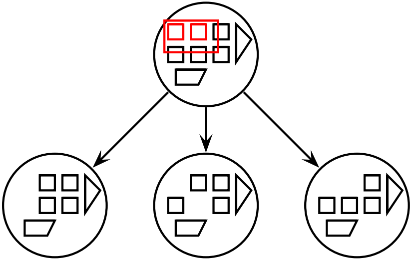

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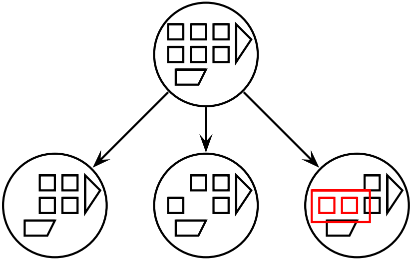

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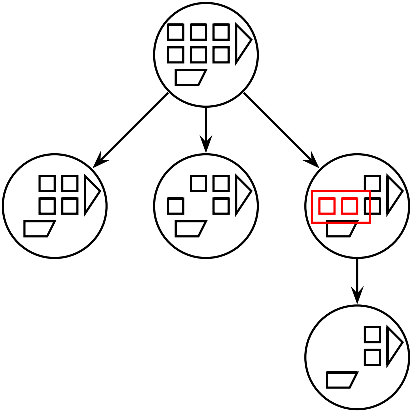

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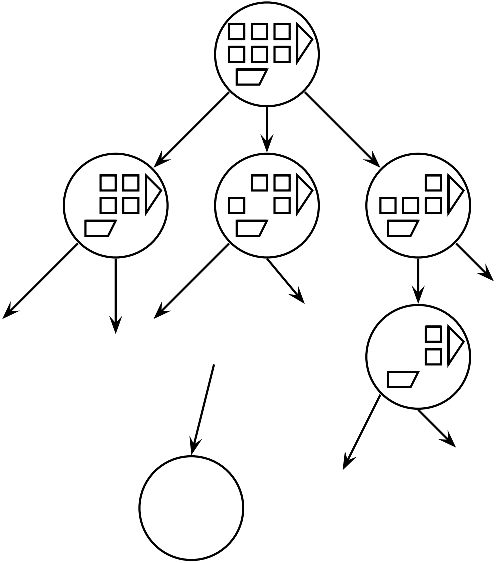

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Goal: find shortest path

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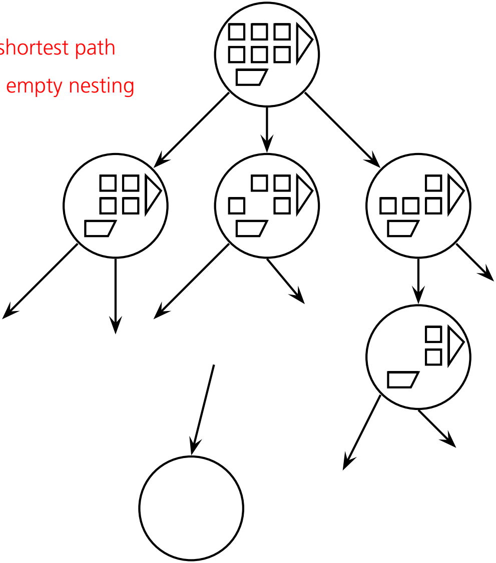

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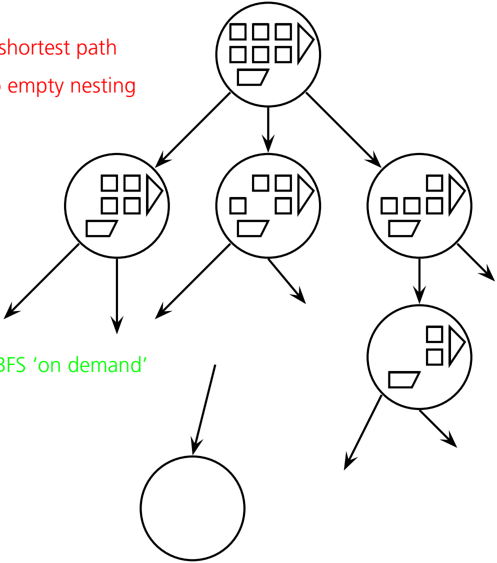


Solution: BFS 'on demand'

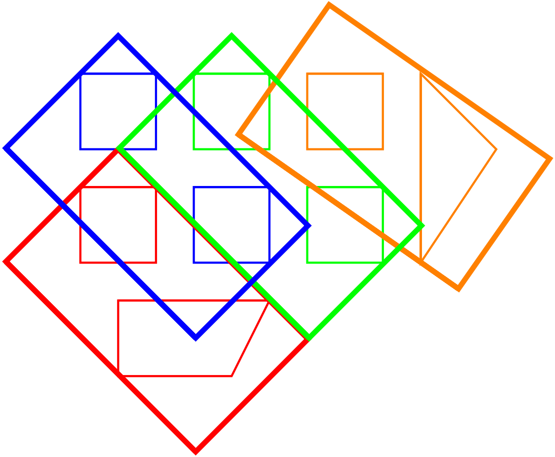
Computation of Candidate Rectangles

How to compute the candidate rectangles?

Two approaches:

- vertex-oriented approach
- polygon-oriented approach

Vertex-Oriented Approach - Alignment of a Rectangle



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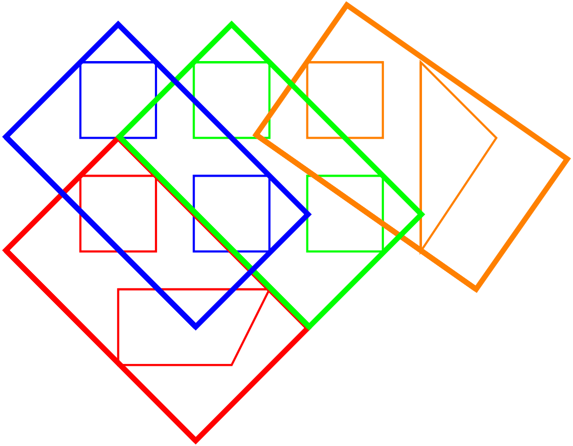

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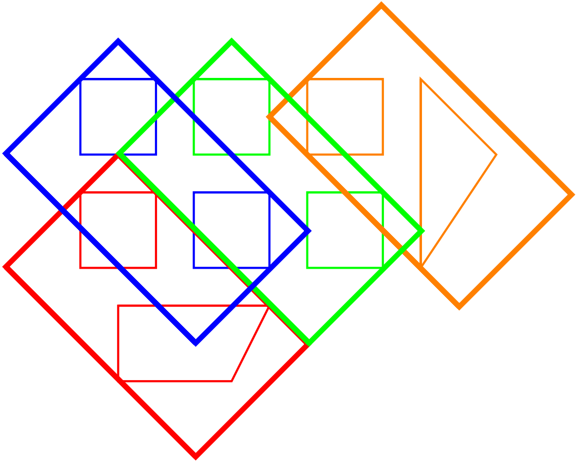

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Observation: Every rectangle R of an optimal cover can be replaced by a rectangle where three vertices lie on at least two different adjacent sides of R, without destroying optimality (degenerated cases possible but easy to handle).

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- determine the set of completely contained polygons
- keep all rectangles containing completely an $\subseteq$-maximal set of polygons including the pivot polygon (modulo cover equivalent rectangles)

Polygon-Oriented Approach

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- abstract from rectangles and
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- by backtracking
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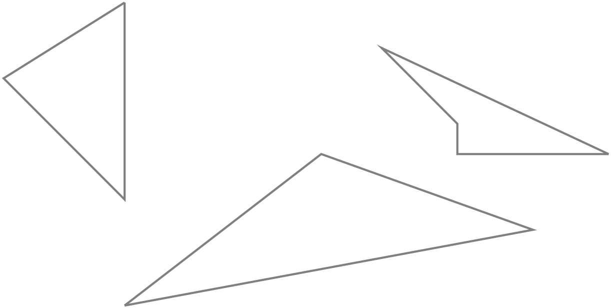
- for a suitable rotational part the translational part is easy to determine
- here: rotate polygon set instead of rectangle for better understanding

Polygon-Oriented Approach - Convex Hull



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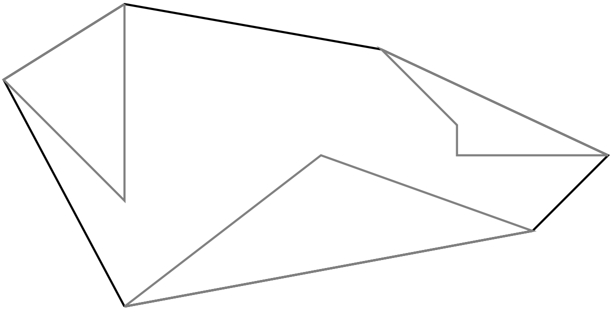

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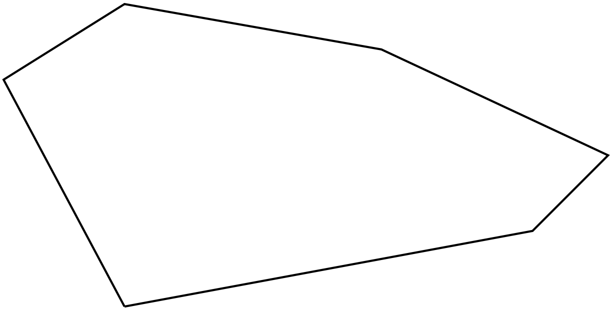
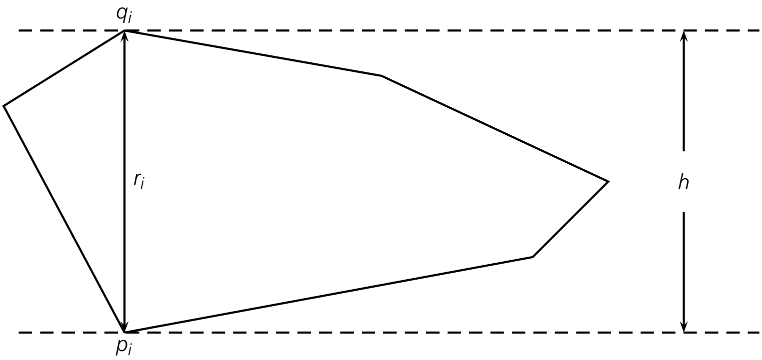


Polygon-Oriented Approach - Determining the Height



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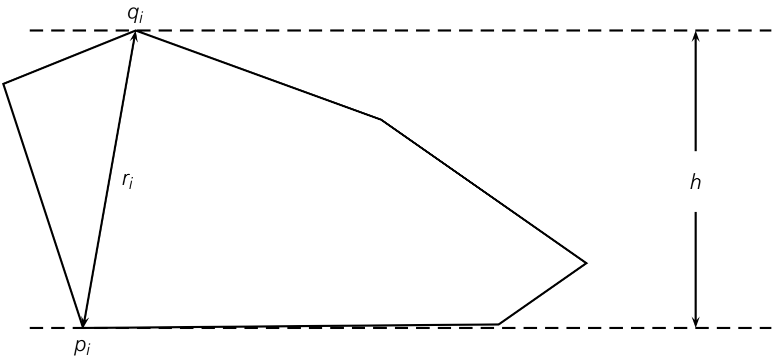

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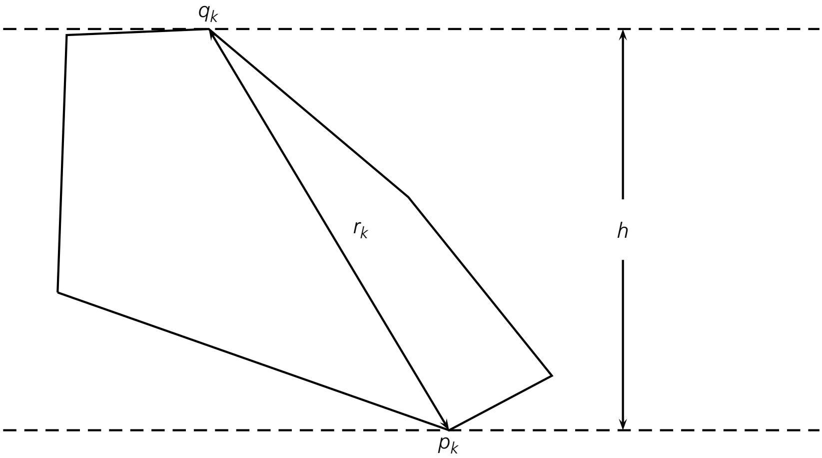

Polygon-Oriented Approach - Intersecting Height and Width Function

- height representable as piecewise defined sine function
- width shifted by $\frac{\pi}{2}$
- determine all intervals where height of the rotated polygon is less-equal to gripper height
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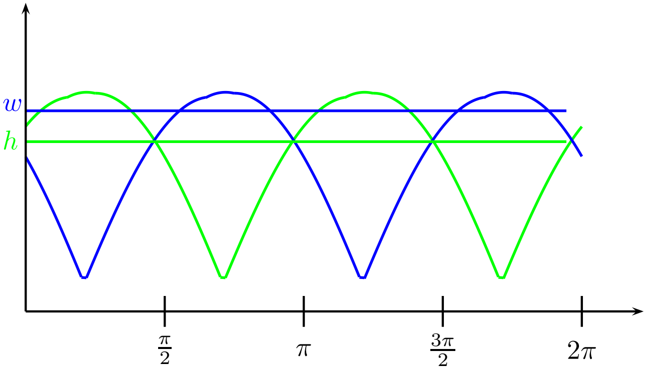
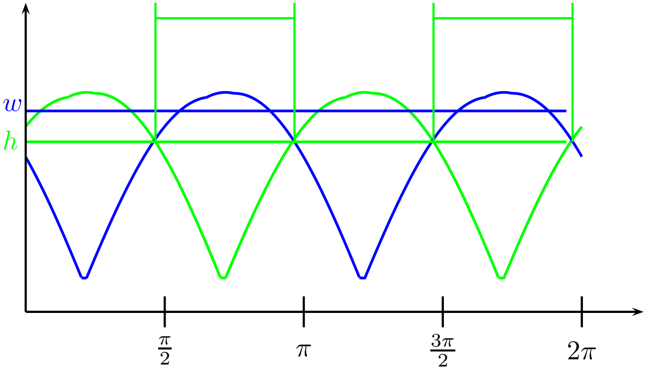
- intersect admissible height- and width intervals
- interval construction in linear time by means of rotating calipers
- intersection in linear time by means of sweep line
- linear overall running time

Polygon-Oriented Approach - Graphic Representation



Polygon-Oriented Approach - Height Intervals



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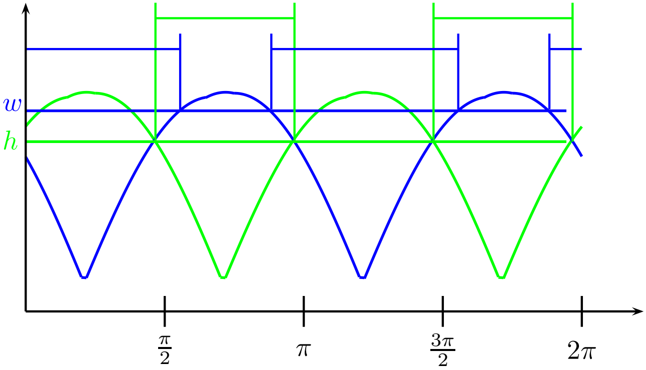
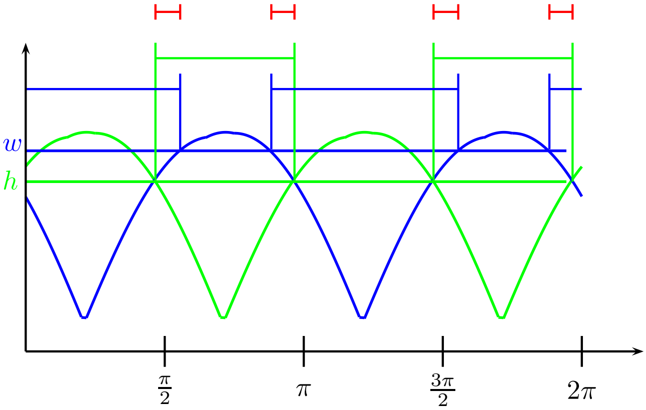


Polygon-Oriented Approach - Suitable Intervals



Running Time Estimations

Notation: $\|\mathbf{P}\|$ denotes overall number of vertices (clearly, $|\mathbf{P}|$ denotes number of polygons)

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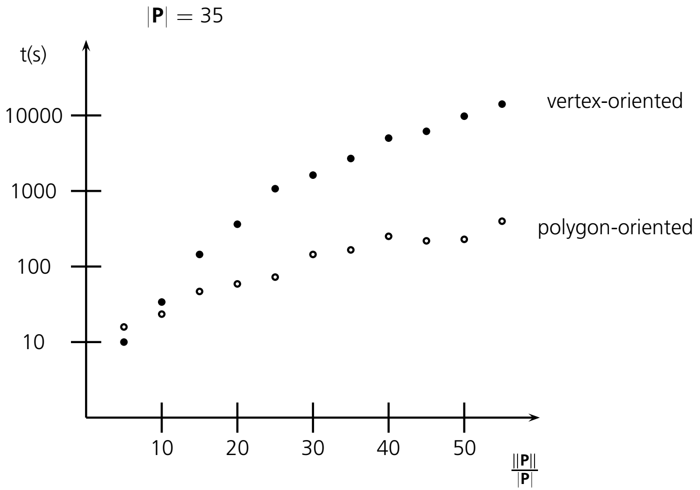
- polygon-oriented approach needs to examine $2^{|\mathbf{P}|}$ combinations of polygons
- but running time linear in $||\mathbf{P}||$
- well-suited for instances with high number of vertices and low number of polygons

Experimental Results



Experimental Results

$\lvert \mathbf{P} \rvert$	$\frac{\lVert \mathbf{P} \rVert}{\lvert \mathbf{P} \rvert}$	polygon-oriented		vertex-oriented			
		μ	c_V	μ	c_V	$\left(\frac{\lVert \mathbf{P}_n \rVert}{\lVert \mathbf{P}_{n-1} \rVert}\right)^3$	$\frac{\mu_n}{\mu_{n-1}}$
25	5	1.5	0.28	1.3	0.21	-	-
25	10	1.3	0.34	9.0	0.2	8	6.9
25	15	1.1	0.28	32	0.15	3.4	3.6
25	20	1.8	0.44	78	0.21	2.4	2.4
25	25	2.0	0.49	160	0.16	2.0	2.1
25	30	1.3	0.17	290	0.13	1.7	1.8
25	35	1.4	0.34	540	0.26	1.6	1.9
25	40	1.6	0.21	760	0.21	1.5	1.4
25	45	2.5	0.40	960	0.22	1.4	1.3
25	50	3.2	0.60	1300	0.10	1.4	1.4
25	55	2.9	0.29	1700	0.27	1.3	1.3

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polygon-oriented			
$\|\|\mathbf{P}\|\|$	$\|\mathbf{P}\|$	μ	$\frac{\mu_n}{\mu_{n-1}}$
2000	5	0.021	-
2000	10	0.064	3.0
2000	15	0.2	3.1
2000	20	1.0	5.0
2000	25	5.2	5.2
2000	30	15	2.9
2000	35	49	3.3
2000	40	400	8.2

vertex-oriented			
$\|\|\mathbf{P}\|\|$	$\|\mathbf{P}\|$	μ	$\frac{\mu_n}{\mu_{n-1}}$
500	5	61	-
500	10	52	0.9 (!)
500	15	48	0.9 (!)
500	20	70	1.5
500	25	75	1.1
500	30	84	1.1
500	35	110	1.3
500	40	120	1.1

Future Work

- parallelizing the search
- approximation for larger instances
- consider other gripper shapes
- workspace limitations of the gripper

Acknowledgments

Thanks to

- Torben Hagerup
- Christian Rähtz
- Lev Sorokin

Covering Questions

Spørgsmål?

ερωτήσεις?

שאלות?

Questions?

Fragen?

Вопросов?

Pitanja?

Kérdések?

Perguntas?

Domande?

¿Preguntas?

Pytania?

Vraagen?

Sorulur?

Întrebari?